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THE FUNDAMENTAL HYDRODYNAMICAL EQUATIONS AND SHOCK CONDITIONS FOR GASES*

T. Y. Thomas

This paper is concerned primarily with the mathematical structure of the derivation of the fundamental differential equations of fluid dynamics and the conditions for shock waves in gases. No attempt is made to treat the existence theoretic aspects of the problem or to discuss underlying physical ideas.

I. VISCOUS AND COMPRESSIBLE FLUIDS

1. *Fundamental dynamical assumptions.* A fluid (liquid or gas) will be considered from our present standpoint as forming a continuous medium. Let the motion of the fluid be represented by equations of the form $x^a = f^a(\bar{x}, t)$ where the x^a and $\bar{x}^a$ are coordinates of the same rectangular system and t denotes the time. A particle, initially at the point $\bar{x}$, will be located at the point x at time t in accordance with these relations. It will be assumed that the equations $x^a = f^a(\bar{x}, t)$ have a unique inverse at any time t and it suffices for our purpose to suppose that the functions $f^a(\bar{x}, t)$ are continuous and have continuous partial derivatives through the third order with respect to the initial coordinates $\bar{x}^a$ and the time t. The velocity components $u^a(x, t)$ are then defined in the usual manner by the equations $u^a = \partial f^a/\partial t$ and these quantities will be continuous in x^a and t and will have continuous derivatives through the second order with respect to these variables.

Now consider a finite volume V of the fluid and denote by S the boundary or surface of V. We suppose that V is a continuous (1,1) map of a cube and that S is a closed regular surface[1]; these conditions will be retained under motions of the fluid particles by which V and S are composed in consequence of the above assumptions on the functions f^a. All volumes and their boundaries in the following discussion are assumed implicitly to be of this type. We make the following dynamical assumptions.

(α) The rate of change of the momentum of V parallel to any fixed direction is equal to the component parallel to this direction of the total external forces acting on V.

(β) The rate of change of the angular momentum of V about any fixed line is equal to the moment of the external forces acting on V about

* This exposition was written for the benefit of students attending the applied mathematics seminar at Indiana University and may be of interest to others. It was prepared under Navy Contract N6onr—180, Task Order No. V, with Indiana University.

this line.

We assume also the *law of the conservation of mass* which states that the mass of the moving volume V remains constant. This condition can be expressed by writing

$$(1)\quad \frac{d}{dt}\int_V \rho\, dV = 0,$$

where ρ is the *density* or mass per unit volume at points of the fluid. In the case of compressible fluids for which ρ is not constant the above assumptions must be supplemented by an *equation of state;* this consists usually of a relation between the density and *pressure* (defined in §6).

The assumptions (α) and (β) are fully satisfied if the conditions involved are expressed for each of three fixed and mutually perpendicular directions in space. We take these perpendicular directions to be the axes of the rectangular coordinnte system x to which the position of the fluid particles is referred. Relative to these coordinates let X^{α} be the components of an external or applied force per unit mass acting in V. Also let T^{α} be the components of a surface force per unit area over the surface S. The momentum per unit volume in the fluid will be given by ρu^{α}. Finally we need an expression for the moment of a force F about the coordinate axes. But the moment of a force F about the x^{α} axis is given by $M_{\alpha} = e_{\alpha\beta\gamma}\, x^{\beta}F^{\gamma}$ where F^{γ} are the components of F and the quantities $e_{\alpha\beta\gamma}$ are defined by the following two requirements, *first* $e_{123} = 1$ and *second* the $e_{\alpha\beta\gamma}$ are skew-symmetric. Using these designations, conditions (α) and (β) have their formal expression in the following two sets of equations

$$(2)\quad \frac{d}{dt}\int_V \rho u^{\alpha} dV = \int_S T^{\alpha} dS + \int_V \rho X^{\alpha} dV,$$

$$(3)\quad \frac{d}{dt}\int_V e_{\alpha\beta\gamma}\, x^{\beta} u^{\gamma} \rho\, dV = \int_S e_{\alpha\beta\gamma}\, x^{\beta} T^{\gamma} dS + \int_V e_{\alpha\beta\gamma}\, x^{\beta} X^{\gamma} \rho\, dV.$$

In the following it will be supposed that $X = 0$, *i.e.*, no applied forces act on the fluid. Also it is assumed, preceeding §18, that all functions of the variables x^{α} and t which enter into our discussion are continuous and have continuous partial derivatives for all values of these variables under consideration. Finally, we remark that covariant and contravariant indices have been interchanged at will as is permissable since we deal exclusively with rectangular coordinate systems. Repeated indices in any term are to be summed in accordance with the usual convention.

2. *Differentiation of integrals with variable boundaries.* Let $V(t)$

be a volume in the x space which may vary with the time t and let S denote the surface of V. It can be shown that

$$\text{(4)} \qquad \frac{d}{dt}\int_V f(x,t)\,dV = \int_V \frac{\partial f}{\partial t}\,dV + \int_S fG\,dS,$$

where G is the component of the velocity of the surface S along the outward normal to S. In the particular case that the variation in V is determined by the motion of the fluid particles we have $G = u_a\nu^a$ where ν^a are the components of the unit normal vector directed outward from S.

3. *Equation of continuity*. Taking $f = \rho$ and $G = u_a\nu^a$ the application of (4) to the condition (1) yields,

$$\int_V \frac{\partial\rho}{\partial t}dV + \int_S \rho u_a\nu^a dS = \int_V \frac{\partial\rho}{\partial t}dV + \int_V (\rho u_a)_{,a}\,dV = 0\,.$$

Since V is arbitrary it follows that,

$$\text{(5)} \qquad \frac{\partial\rho}{\partial t} + (\rho u_a)_{,a} \equiv \frac{d\rho}{dt} + \rho u_{a,a} = 0\,.$$

Equation (5) is called the *equation of continuity;* it is equivalent to the condition (1) for arbitrary volumes V.

A useful result can be derived from (4) when $G = u_a\nu^a$ and the equation of continuity is satisfied. We have

$$\int_S fu_a\nu^a dS = \int_V (fu_a)_{,a}\,dV = \int_V f_{,a}\,u_a dV + \int_V fu_{a,a}\,dV =$$

$$\int_V f_{,a}\,u_a dV - \int_V \frac{f}{\rho}\frac{d\rho}{dt}\,dV\,.$$

Hence, $$\frac{d}{dt}\int_V f\,dV = \int_V \frac{df}{dt}\,dV - \int_V \frac{f}{\rho}\frac{d\rho}{dt}\,dV.$$

Then, replacing f by ρf, we obtain,

$$\text{(6)} \qquad \frac{d}{dt}\int_V \rho f\,dV = \int_V \rho\,\frac{df}{dt}\,dV.$$

This equation is satisfied in a fluid when the variation in V is produced by the motion of the fluid particles.

4. *The stress tensor.* It can readily be shown[2] in consequence of the condition (2) that the above vector T representing the force per unit area or *stress* over the surface S has components of the form

$$\text{(7)} \qquad T_a = \tau_{a\beta}\nu^\beta,$$

where the $\tau_{\alpha\beta}$ are functions of position and time. From the vector character of T and ν and the relation (7) we infer that the quantities $\tau_{\alpha\beta}$ are the components of a tensor under coordinate transformations. It will now follow from (2) and (3) that the tensor τ is symmetric.

Using (6) equations (2) and (3) become

$$\int_V \rho \frac{du^\alpha}{dt} dV = \int_S \tau_{\alpha\beta} \nu^\beta dS = \int_V \tau_{\alpha\beta,\beta}\, dV,$$

$$\int_V \rho e_{\alpha\beta\gamma} \frac{d}{dt}(x^\beta u^\gamma) dV = \int_S e_{\alpha\beta\gamma} x^\beta \tau_{\gamma\delta}\, \nu^\delta dS = \int_V (e_{\alpha\beta\gamma} x^\beta \tau_{\gamma\delta})_{,\delta}\, dV.$$

When we carry out the differentiations indicated in the second of these equations the two sets of equations can be written

$$(8) \qquad \int_V [\rho \frac{du^\alpha}{dt} - \tau_{\alpha\beta,\beta}] dV = 0,$$

$$(9) \qquad \int_V e_{\alpha\beta\gamma} x^\beta (\rho \frac{du^\alpha}{dt} - \tau_{\gamma\delta,\delta})\, dV + \int_V e_{\alpha\beta\gamma} \tau_{\beta\gamma}\, dV = 0.$$

Then, since V is arbitrary, we have

$$(10) \qquad \rho \frac{du^\alpha}{dt} = \tau_{\alpha\beta,\beta}$$

from (8). But from (9) and (10) it now follows that $e_{\alpha\beta\gamma}\tau_{\beta\gamma} = 0$ and these conditions imply the symmetry of the quantities $\tau_{\alpha\beta}$. *Hence we see that* (2) *and* (3) *can be replaced by the equations* (10) *with the condition that the tensor* τ *be symmetric.*

5. *The distortion tensor.* Suppose that in a small interval of time dt the point x is displaced into the point $\bar{x}$ as a result of the flow. Then to the first order of approximation the displacement in the medium is given by

$$(11) \qquad \bar{x}^\alpha = x^\alpha + u^\alpha(x,t)dt.$$

It can be shown[3] that the change $\delta\phi$ of an angle ϕ produced by the displacement (11) is given by

$$(12) \qquad \sin\phi\ \delta\phi = [(D_{\alpha\beta}\nu_1^\alpha \nu_1^\beta + D_{\alpha\beta}\nu_2^\alpha \nu_2^\beta)\cos\phi - 2D_{\alpha\beta}\nu_1^\alpha \nu_2^\beta]dt,$$

to the first approximation, where ν_1 and ν_2 are unit vectors determining the angle ϕ and

$$(13)\qquad D_{\alpha\beta} = e_{\alpha\beta} - \frac{\theta}{3}\,\delta_{\alpha\beta}, \quad [e_{\alpha\beta} = \tfrac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}), \quad \theta = u_{\sigma,\sigma}].$$

The tensor D defined by (13) will be called the *distortion tensor.*

If $\delta\phi = 0$ for arbitrary directions ν_1 and ν_2 any volume V will be displaced into a volume V without change of shape, or as we shall say, *without distortion*. It can easily be shown that the necessary and sufficient condition for the displacement (11) to be without distortion to the first approximation is given by the vanishing of the distortion tensor[4]. Hence we see from (12) that the tensor D at any point P can be considered to give a measure of the *rate of distortion*, due to the flow, of any configuration in the immediate neighborhood of P.

6. *The viscosity tensor.* The viscosity tensor V is defined in terms of the stress tensor τ as follows

$$(14)\qquad V_{\alpha\beta} = \tau_{\alpha\beta} + p\delta_{\alpha\beta},$$

where p is a scalar function given by $p = -\tau_{\alpha\alpha}/3$. The function p is called the *pressure*.

By definition a fluid is called *ideal* if, and only if, the viscosity tensor vanishes. For ideal fluids we therefore have $\tau_{\alpha\beta} = -p\delta_{\alpha\beta}$ and hence $T_\alpha = -p\nu_\alpha$; hence the force T is normal to every surface element. In the general case the tangential component to a surface S of the force T is called a *shearing force*. It follows that for ideal fluids the shearing force is zero over arbitrary surface elements and conversely it is easily seen that this condition implies the vanishing of the viscosity tensor. Hence an ideal fluid can also be defined as one for which the shearing force vanishes over any surface element independently of its orientation.

7. *Relation between the distortion and viscosity tensors.* We now make the fundamental assumption that the components of the viscosity tensor are linear and homogeneous functions (independent of the time t) of the components of the distortion tensor. Thus

$$(15)\qquad V_{\alpha\beta} = C^{\sigma\eta}_{\alpha\beta}(x)\,D_{\sigma\eta}.$$

Since V and D are tensors these relations will be invariant under coordinate transformations if we define the quantities $C^{\sigma\eta}_{\alpha\beta}$ to be the components of a tensor. The tensor C so defined, can be taken to be symmetric in its two upper and also in its two lower indices due to the symmetry property of the tensors D and V.

8. *Homogeneous and isotropic fluids.* The equations (15) will be specialized by further assumptions. First we assume that the fluid is *homogeneous*. This means that the functions $C^{\sigma\eta}_{\alpha\beta}$ are unchanged by an arbitrary translation of the rectangular coordinate axes. Then under the

translation $y^a = x^a + c^a$ we have

$$C^{\sigma\eta}_{a\beta}(y) = C^{\sigma\eta}_{a\beta}(x) = C^{\sigma\eta}_{a\beta}(y-c).$$

Since the c^a are arbitrary constants we see from the equality of the first and last members of these relations that the values of the C's are independent of the coordinates; conversely if the C's are constants relative to rectangular coordinates the fluid is homogeneous. *Hence the requirement that the fluid is homogeneous means that the coefficients C in (15) are constants in a rectangular coordinate system.*

Next assume that the fluid is *isotropic*. This means that the values of the C's at any point must remain unchanged under proper orthogonal rotations of the coordinates about this point. A tensor possessing this property is called an *isotropic tensor*. It can be shown that the most general (symmetric) isotropic tensor C has components given by

$$C^{\sigma\eta}_{a\beta} = \lambda\delta^{\sigma\eta}\,\delta_{a\beta} + \mu(\delta^{\sigma}_{a}\,\delta^{\eta}_{\beta} + \delta^{\sigma}_{\beta}\,\delta^{\eta}_{a}), \tag{16}$$

relative to rectangular coordinates, where λ and μ are scalars[5]. When the fluid is both homogeneous and isotropic λ and μ will be constants.

Substituting the values of the C's from (16) into (15) we have

$$V_{a\beta} = \lambda D_{\sigma\sigma}\,\delta_{a\beta} + 2\mu D_{a\beta}.$$

When we further substitute for the $D_{a\beta}$ and the $V_{a\beta}$ the values of these components given by (13) and (14) we obtain

$$\tau_{a\beta} = -p\delta_{a\beta} - \frac{2}{3}\,\mu\theta\delta_{a\beta} + \mu(u_{a,\beta} + u_{\beta,a}). \tag{17}$$

Hence (17) is the most general expression for the quantities $\tau_{a\beta}$ in a homogeneous and isotropic fluid (for which the fundamental relation (15) is assumed). In (17) the scalar μ is a constant and is referred to as the *viscosity constant* or simply the *viscosity* of the fluid. Observe that if we contract the indices a and β in (17) we obtain the previous relation $\tau_{aa} + 3p = 0$ by which the pressure was defined.

9. *The Navier-Stokes equations.* We have seen that the equations (2) and (3) are equivalent to (10) in which the $\tau_{a\beta}$ are symmetric. We now consider the explicit form of (10) when the $\tau_{a\beta}$ are given by

$$\tau_{a\beta,\beta} = -p_{,a} + \frac{\mu}{3}\theta_{,a} + \mu u_{a,\beta\beta}.$$

Substituting this expression for the $\tau_{a\beta,\beta}$ into (10) we obtain

$$\mu u_{a,\sigma\sigma} = p_{,a} + \rho\frac{\partial u_a}{\partial t} + \rho u_\sigma\, u_{a,\sigma} - \frac{\mu}{3}\,\theta_{,a} \tag{18}$$

where $\theta = u_{\sigma,\sigma}$ and μ is taken to be a non-negative constant (the viscosity). *Equations (18) are the equations of motion of a homogeneous, isotropic viscous fluid;* they are usually referrec to as the *Navier-Stokes equations*. These equations together with the equation of continuity (5) and an equation of state (§1) furnish us with the theoretic means of solving a rather general class of problems of fluid motion.

10. *Special motions.* It is customary in practical applications to narrow the class of admissable motions by making certain additional simplifying assumptions which can be justified in greater or less degree by the nature of the problem under consideration. Thus in certain circumstances we can assume that $\mu = 0$ and/or ρ = const. Also it is frequently assumed that the velocity vector is the gradient of a scalar function in consequence of which we can write $u_\alpha = \psi_{,\alpha}$; in this case it is customary to say that the velocity admits a potential ψ. It is usually simpler, moreover, to confine our attention to *steady problems*, i.e., those problems of fluid motion for which the pressure, density and velocity are independent of the time. Of the various possibilities we note here that the differential equations for the steady motion of *compressible fluids* (gases) in the absence of viscosity are the following

$$(19) \qquad p_{,\alpha} + \rho u_\sigma u_{\alpha,\alpha} = 0 , \quad (\rho u_\sigma)_{,\sigma} = 0 .$$

If we suppose, in addition, that ρ = const. and that the velocity admits a potential ψ the equations (19) become

$$(20) \qquad p_{,\alpha} + \tfrac{1}{2}(\rho u_\sigma u_\sigma)_{,\alpha} = 0 , \quad \psi_{,\sigma\sigma} = 0 .$$

These are the differential equations of classical hydrodynamics. But the first set of (20) can be integrated and hence can be regarded essentially as an equation defining p when the velocity and density are known; in this case the problem of determining the fluid motion reduces to the solution of the Laplace equation $\Delta\psi = 0$ subject to the given boundary conditions.

II. IDEAL GASES

11. *The energy equation.* The work done per unit time on a moving volume V of the fluid by the stresses acting over its surface S is given by

$$W = \int_S \tau_{\alpha\beta} u^\alpha \nu^\beta dS = \int_V (\tau_{\alpha\beta} u^\alpha)_{,\beta} dV$$

$$= \int_V \tau_{\alpha\beta,\beta} u^\alpha dV + \int_V \tau_{\alpha\beta} u^\alpha_{,\beta} dV.$$

Making the substitutions (10) and (17) and using (6) this expression

for W can be put in the form

$$W = \frac{d}{dt}\int_V \tfrac{1}{2}\rho u_\sigma u_\sigma dV - \int_V p u_{\sigma,\sigma}\, dV + \int_V \Phi dV, \tag{21}$$

where Φ is a non-negative function given by

$$\Phi = 2\mu D_{\alpha\beta}\, D_{\alpha\beta}\,.$$

Equation (21) is called the *energy equation* and the function Φ is referred to as the *dissipation function*. The first term in the right member of (21) gives the rate of increase of the kinetic energy of the moving volume V. The second term gives the rate at which work is done by the pressure in changing the volume of V. The remaining term represents the rate at which work is done on the volume against the viscous forces; this work is lost as mechanical energy and must therefore appear in the form of heat developed in the volume V.

12. *The ideal gas*. By an ideal gas is meant a gas which satisfies the equation

$$\frac{p}{\rho} = \frac{R}{m}T, \tag{22}$$

where T is the absolute temperature, m the molecular weight and R an absolute constant which is the same for all gases. If c_p and c_v are the specific heats of the gas at constant pressure and constant volume respectively, then between the quantities R, m, c_p and c_v we have the relation

$$\frac{R}{m} = J(c_p - c_v), \tag{23}$$

where J is the mechanical equivalent of heat. *It is assumed that c_p and c_v are constant for an ideal gas.* Defining the constant γ as the ratio c_p/c_v the above relations (22) and (23) can be written

$$\frac{p}{\rho} = Jc_v(\gamma - 1)\,T, \quad \frac{R}{m} = Jc_v(\gamma - 1). \tag{24}$$

13. *Thermal conductivity*. The total energy of a gas consists of its kinetic and internal energies. For an ideal gas it can be shown by thermodynamical considerations that the internal energy per unit mass is expressed by Jc_vT + const. where the constant depends on the selection of the zero point of energy. Hence the total energy E per unit mass is given by

$$E = \tfrac{1}{2}u_\sigma u_\sigma + Jc_vT + \text{const.} \tag{25}$$

Using (25) the expression (21) for W can be given the form

$$W = \frac{d}{dt}\int_V \rho E dV - Jc_v \int_V \rho \frac{dT}{dt}\, dV - \int_V p u_{\sigma,\sigma}\, dV + \int_V \Phi dV, \tag{26}$$

since the constant appearing in (25) gives no contribution to W in consequence of the equation of continuity.

Now let W' denote the rate at which heat is conducted into the moving volume V through its surface S. The value of W' is determined on the assumption that the rate of conduction of heat into V through an element of area dS is given by $Jk(dT/dn)dS$ where k is a constant and n denotes distance measured in the direction of the outward normal ν to S. The constant k is called the *thermal conductivity* of the gas. Hence

$$W' = Jk \int_S T_{,a}\, \nu^a\, dS = Jk \int_V T_{,aa}\, dV. \tag{27}$$

Now $W + W'$ is equal to the rate of change of the total energy of V so that

$$\frac{d}{dt}\int_V \rho E dV = W + W'. \tag{28}$$

Hence substituting into this relation the expressions for W and W' given by (26) and (27) we find

$$Jc_v \int_V \rho \frac{dT}{dt}\, dV + \int_V p u_{\sigma,\sigma}\, dV - \int_V \Phi dV - Jk \int_V T_{,\sigma\sigma}\, dV = 0,$$

or, since the volume V is arbitrary, we have

$$Jc_v\, \rho \frac{dT}{dt} + p u_{\sigma,\sigma} = JkT_{,\sigma\sigma} + \Phi. \tag{29}$$

This is the equation governing the distribution of temperature in an ideal gas. From the equations (22) and (23) we have

$$Jc_v T = Jc_p T - \frac{RT}{m} = Jc_p T - \frac{p}{\rho}.$$

Hence differentiating with respect to t and substituting in the left member of (29) we obtain

$$Jc_p\, \rho \frac{dT}{dt} - \frac{dp}{dt} = JkT_{,\sigma\sigma} + \Phi, \tag{30}$$

when use is made of the equation of continuity. Equation (30) is an alternate form of (29). *If the viscosity μ and thermal conductivity k*

are assumed to vanish equations (29) *and* (30) *become*

$$(31) \qquad Jc_v \, \rho \frac{dT}{dt} + p u_{\sigma,\sigma} = 0, \qquad Jc_p \, \rho \frac{dT}{dt} - \frac{dp}{dt} = 0.$$

14. *Entropy.* From the discussion in §11 and §13 we see that the rate of increase of the heat content of the moving volume is given by

$$Q = \int_S JkT_{,a} \nu^a \, dS + \int_V \phi dV = Jk \int_V T_{,\sigma\sigma} \, dV + \int_V \phi dV.$$

Hence $JkT_{,\sigma\sigma} + \phi$ gives the rate of increase of heat per unit volume at points in the gas. But this expression is equal to the left member of (29) and hence when we divide the left member of (29) by ρ we obtain an expression for the rate of increase of heat per unit mass. This latter expression can now be modified as follows. We have

$$\begin{aligned} Jc_v \frac{dT}{dt} + \frac{p}{\rho} u_{\sigma,\sigma} &= Jc_v \left[\frac{dT}{dt} + (\gamma - 1) T u_{\sigma,\sigma}\right], \\ &= Jc_v \left[\frac{dT}{dt} - (\gamma - 1) \frac{T}{\rho} \frac{d\rho}{dt}\right], \\ &= Jc_v T \left[\frac{d \log T}{dt} - (\gamma - 1) \frac{d \log \rho}{dt}\right], \\ &= Jc_v T \left[\frac{d}{dt} \log\left(\frac{p}{\rho}\right) - (\gamma - 1) \frac{d \log \rho}{dt}\right], \\ &= Jc_v T \frac{d}{dt} \log\left(\frac{p}{\rho^\gamma}\right). \end{aligned}$$

We now define a function σ by the equation

$$(32) \qquad \sigma = Jc_v \log\left(\frac{p}{\rho^\gamma}\right) + \text{const.},$$

where the constant is absolute, i.e. independent of position and time. Hence $T\, d\sigma/dt$ is the rate of increase of heat per unit mass. The function σ is called the *entropy per unit mass* or simply the *entropy* of the gas and for any particular gas it is given, by definition, to within an additive constant. Solving (32) we can also write

$$(33) \qquad p = N(\sigma)\rho^\gamma, \qquad N(\sigma) = C e^{\sigma / Jc_v},$$

where C is an absolute constant for any gas.

If the thermal conductivity k and viscosity μ are both equal to

zero it follows from (29) that $d\sigma/dt = 0$. *Hence in an ideal gas with conductivity and viscosity zero the entropy is constant following each particle.* In particular if the motion is steady the entropy is constant along each stream line; this implies that the function $N(\sigma)$ in the relation (32) is constant along each stream line although this function can vary from stream line to stream line. When the motion is such that N is constant throughout the gas we say that the gas satisfies the *adiabatic condition* or that the flow is *adiabatic*. The assumption that the flow is adiabatic results in considerable simplification of the mathematical analysis but is, strictly speaking, an untenable hypothesis except in the case of certain especially selected problems.

15. *An equivalent of the entropy condition.* Assuming $k = \mu = 0$ so that $d\sigma/dt = 0$ let us differentiate (33) totally with respect to t; on making use of the equation of continuity (5) and the equations of motion (18) with $\mu = 0$ the resulting equation can be written

$$\frac{\partial p}{\partial t} - \rho u_\sigma \frac{\partial u_\sigma}{\partial t} - \rho u_{\alpha,\beta} u_\alpha u_\beta + \gamma p u_{\sigma,\sigma} = 0. \tag{34}$$

Conversely from (34), the equation of continuity, and the equations of motion with $\mu = 0$ we can show that $d\sigma/dt = 0$ in the relation (33). *Hence in an ideal gas with thermal conductivity and viscosity equal to zero the condition $d\sigma/dt = 0$ is fully expressed by* (34). In case the motion is steady the condition $d\sigma/dt = 0$ is given by

$$\rho u_{\alpha,\beta} u_\alpha u_\beta - \gamma p u_{\sigma,\sigma} = 0. \tag{35}$$

16. *The differential equations for ideal gases.* The complete set of differential equations by which the motion of an ideal gas is determined is given by the equation of continuity (5), the equations of motion (18) and the equation (29) for the distribution of temperature in the gas; in addition there is an *equation of state* (22) which was originally introduced as the defining condition of the ideal gas. It is of course possible to eliminate the temperature T between (22) and (29). When this is done we have in place of (29) a differential equation in the quantities p, ρ and u_α; the relation (22) then appears as an extraneous equation defining the temperature T. *In particular the steady motion of an ideal gas without viscosity and thermal conductivity is governed by the differential equations,*

$$(\rho u_\sigma)_{,\sigma} = 0, \tag{36a}$$

$$p_{,\alpha} + \rho u_\sigma u_{\alpha,\sigma} = 0, \tag{36b}$$

$$\rho u_{\alpha,\beta} u_\alpha u_\beta - \gamma p u_{\sigma,\sigma} = 0. \tag{36c}$$

As we saw in §15 the last of these equations is equivalent to the entropy condition $d\sigma/dt = 0$ and hence from §14 it is equivalent to (29) with $k = \mu = 0$ or to either of the equations (31).

17. *The Bernoulli equation.* Multiplying (36b) by u_a and summing on the repeated index a we obtain

$$\frac{dp}{dt} + \tfrac{1}{2}\rho \frac{d}{dt}(u_a u_a) = 0.$$

Dividing this equation by ρ and making use of (33) in which σ = const. along the stream lines we find that

$$\frac{\gamma}{\gamma-1}\frac{p}{\rho} + \tfrac{1}{2}u_\sigma u_\sigma = \text{const.}, \tag{37}$$

where the constant depends on the stream line. Equation (37) is the *Bernoulli equation. It holds for the steady motion of an ideal gas for which the viscosity and thermal conductivity are zero.*

When we assume that the velocity admits a potential (§10) equations (36b) can be written

$$\frac{p_{,a}}{\rho} + \tfrac{1}{2}(u_\sigma u_\sigma)_{,a} = 0.$$

If, in addition, the flow is adiabatic (§14) the pressure p in these equations can be eliminated by the substitution (33) and the resulting equations can be integrated. This leads to an equation of the form (37) in which the constant in the right member is independent of the stream line. To distinguish between these two cases the equation (37) with constant depending on the stream line is sometimes referred to as the *weak* Bernoulli equation and when this constant is absolute, i.e., independent of the stream line, the equation is called the *strong* Bernoulli equation.

III SHOCK WAVES IN IDEAL GASES

18. *Singular surfaces.* Consider a moving surface $\Sigma(t)$ such that each point of $\Sigma(t)$ for t fixed, lies in a regular surface element[1] and let $f(x,t)$ be a function which is discontinuous in the variables x^a at all points of $\Sigma(t)$. For points x^a not on Σ at any time t we assume that $f(x,t)$ is continuously differentiable in all variables; it is assumed also that $f(x,t)$ and its derivatives exist and are continuous on each side of Σ. *The surface $\Sigma(t)$ will be said to be singular relative to the function $f(x,t)$.*

Now let V be any moving volume in the fluid which is divided by the above surface Σ into two volumes V_1 and V_2. Let S be the surface

of V and denote by S_1 and S_2 the portions of S which form parts of the boundaries of V_1 and V_2 respectively; the remaining part of the boundary of V_1 and V_2 will then be furnished by the surface Σ. As in the foregoing discussion we suppose that the variation in the volume V is produced by the moving particles of the fluid so that the normal component of the velocity of V at points of its surface S is given by $u_n = u_a \nu^a$ where ν is the outward unit normal vector to S. Let G denote the normal velocity of $\Sigma(t)$ along the outward normal to Σ considered as part of the boundary of V_1; then $-G$ will be the corresponding normal velocity when Σ is taken as a part of the boundary of V_2. Now

$$\frac{d}{dt}\int_V f(x,t)dV = \frac{d}{dt}\int_{V_1} f(x,t)dV + \frac{d}{dt}\int_{V_2} f(x,t)dV. \tag{38}$$

But each term in the right member of this equation can be evaluated in accordance with the formula (4). This gives

$$\frac{d}{dt}\int_{V_1} f(x,t)dV = \int_{V_1}\frac{\partial f}{\partial t}dV + \int_{S_1} fu_n dS + \int_\Sigma f_1 GdS,$$

$$\frac{d}{dt}\int_{V_2} f(x,t)dV = \int_{V_2}\frac{\partial f}{\partial t}dV + \int_{S_2} fu_n dS - \int_\Sigma f_2 GdS,$$

where f_1 is the value of f on the side of Σ bordering V_1 and f_2 the value of this function on the side of Σ bordering V_2; the designation Σ as the area of integration in the last integrals in these equations, as well as in corresponding integrals in following equations, naturally refers to the part of the surface Σ bounding the volumes V_1 and V_2. Hence substituting into (38) the resulting equation can be written

$$\frac{d}{dt}\int_V f(x,t)dV = \int_V \frac{\partial f}{\partial t}dV + \int_S fu_n dS + \int_\Sigma (f_1 - f_2)GdS. \tag{39}$$

19. *Shock waves*. We now assume that the surface $\Sigma(t)$ is singular relative to the pressure p. The quantities ρ and u_a will be assumed to be continuous and differentiable for points x not on $\Sigma(t)$; also these quantities and their derivatives are assumed continuous on each side of $\Sigma(t)$. The moving volume V, which is divided into the volumes V_1 and V_2 of §18, must satisfy all integral conditions previously imposed and which have resulted in the differential equations for the motion of the gas under the hypothesis of complete continuity. These integral conditions are the following: *first*, the law of conservation of mass, *second*, the assumptions (α) and (β) of §1 and *third*, the assumption contained in the equation (28) which led to the differential equation for the distribution of temperature in an ideal gas. By allowing the volume V to shrink to zero we thus obtain certain relations,

called *shock conditions*, between the values of the quantities p, ρ and u_α on the two sides of the surface $\Sigma(t)$. A surface $\Sigma(t)$, singular relative to the pressure p, over which these shock conditions are satisfied is referred to as a *shock wave*. In the following sections we derive the shock conditions for ideal gases for which the viscosity and thermal conductivity are zero.

20. *First shock condition.* For convenience of terminology we shall refer to the side of Σ bordering V_1 as the side 1 and to the side bordering V_2 as the side 2 of Σ. Then from (1) and (39) with $f = \rho(x,t)$ we have

$$\int_V \frac{\partial\rho}{\partial t}\,dV + \int_{S_1} \rho\, u_n dS + \int_{S_2} \rho\, u_n dS + \int_\Sigma (\rho_1 - \rho_2) G dS = 0,$$

where ρ_1 and ρ_2 denote the value of the density along the sides 1 and 2 of Σ. Now let V approach zero at a fixed time t in such a way that in the limit it passes into a finite part Σ_0 of the surface Σ. Then the volume integral in the above equation is of higher order than the surface integrals and can be neglected in comparison with the latter. Also[6]

$$\int_{S_1} \rho\, u_n dS \to - \int_{\Sigma_0} \rho_1\, u_{1n}\, dS,$$

$$\int_{S_2} \rho\, u_n dS \to \int_{\Sigma_0} \rho_2\, u_{2n}\, dS,$$

where u_{1n} and u_{2n} denote the normal components of the fluid velocities on the sides 1 and 2 of Σ along the normal direction from the side 1 to the side 2. Hence we obtain

$$\int_{\Sigma_0} (\rho_1\, u_{1n} - G\rho_1) dS - \int_{\Sigma_0} (\rho_2\, u_{2n} - G\rho_2) dS = 0.$$

Since this condition is independent of the extent of the surface of integration Σ_0, we have

$$\rho_1\,(u_{1n} - G) = \rho_2\,(u_{2n} - G) \tag{40}$$

over $\Sigma(t)$. *This is the first of the shock conditions.* It originates entirely from the principle of the conservation of mass.

21. *Second shock condition.* The assumption (a) of §1 has its expression in the equation (2) in which we suppose the applied force $X = 0$. Assuming the viscosity $\mu = 0$ so that $T_\alpha = -p\nu_\alpha$ the condition (2) thus becomes

$$\frac{d}{dt}\int_V \rho u_\alpha dV + \int_S p\nu_\alpha\, dS = 0. \tag{41}$$

We now allow the volume V to approach zero as stated in §20; it is then easily seen that

$$(42) \qquad \int_S p\nu_a \, dS \to \int_{\Sigma_0} (p_2 - p_1)\nu_a \, dS,$$

where the ν_a in the limit integral are the components of the unit normal directed from the side 1 to the side 2 of Σ. Also applying (39) with $f = \rho u_a$ the first term in (41) becomes

$$\int_V \frac{\partial}{\partial t}(\rho u_a)dV + \int_{S_1} \rho u_a u_n dS + \int_{S_2} \rho u_a u_n dS + \int_\Sigma (\rho_1 u_{1a} - \rho_2 u_{2a})GdS,$$

and this expression, on passage to the limit, yields

$$(43) \qquad \int_{\Sigma_0} \rho_1 u_{1a}(G - u_{1n})dS - \int_{\Sigma_0} \rho_2 u_{2a}(G - u_{2n})dS.$$

Hence substituting (42) and (43) for the two terms comprising the left member of (41) and making use of the first shock condition (40) we are led to the relations

$$(44) \qquad [p]\,\nu_a = \rho_1 (G - u_{1n})\,[u_a],$$

where we have put $[p] = p_2 - p_1$ and $[u_a] = u_{2a} - u_{1a}$. The brackets [ ] are here used to denote the change in a quantity in the passage over the surface Σ. This convention will be adhered to in the following discussion.

Turning now to the assumption (β) of §1 which is expressed by the equation (3) it can be shown that no new relations are obtained beyond those given by (40) and (44). *Hence* (44) *constitutes the second of the shock conditions according to the enumeration of* §19. We emphasize that this shock condition holds for fluids or gases for which the viscosity is equal to zero.

The product $\rho_1(G - u_{1n})$ *cannot vanish at any point of the surface* Σ. For if this quantity would be equal to zero it would follow from (44) that $[p] = 0$ thereby contradicting the condition for the surface Σ to be a shock wave (§19). The fact that the product $\rho_1(G - u_{1n})$ does not vanish, and hence that $\rho_2(G - u_{2n})$ does not vanish by the first shock condition, is used in the following derivations.

Multiplying (44) by ν_a and summing on the repeated index a we can deduce the relations

$$(45) \qquad [u_a] = [u_n]\,\nu_a, \qquad [p] = \rho_1(G - u_{1n})[u_n]$$

by which $[u_a]$ and $[p]$ are expressed in terms of $[u_n]$. Conversely (45) implies (44). *Hence* (45) *is equivalent to the shock condition* (44). It follows from the second equation (45) and the condition $[p] \neq 0$ that

$[u_n]$ can not vanish at points of a shock wave Σ; using this result the first equation (45) implies that not all of the quantities $[u_\alpha]$ can be equal to zero at any point of Σ if this surface is a shock wave.

Again, using (40) and (45) we can deduce the relations

$$(46) \qquad [u_\alpha] = \frac{(G - u_{1n})[\rho]\nu_\alpha}{\rho_2},$$

$$(47) \qquad [p] = \frac{\rho_1 (G - u_{1n})^2 [\rho]}{\rho_2},$$

from which the conditions (40) and (45) can be recovered. *Hence the conditions (46) and (47), expressing $[u_\alpha]$ and $[p]$ in terms of $[\rho]$, are equivalent to the first and second shock conditions.*

Finally observe that if we multiply (44) by λ^α, where λ is a unit vector tangent to Σ, we obtain $[u_\alpha \lambda^\alpha] = 0$. *In other words the tangential components of fluid velocity are continuous across the surface Σ.*

22. *The Rankine - Hugoniot relation.* We now impose the full restrictions mentioned in §19, namely that we are dealing with an ideal gas with viscosity and thermal conductivity equal to zero. Then the quantity W' of §13 vanishes and $\tau_{\alpha\beta} = -p\delta_{\alpha\beta}$; hence substituting for W in (28) this equation becomes

$$(48) \qquad \frac{d}{dt}\int_V \rho E dV + \int_S p u_\alpha \nu^\alpha dS = 0.$$

But from (39) with $f = \rho E$ the first term of this equation can be written

$$\int_V \frac{\partial}{\partial t}(\rho E)dV + \int_S \rho E u_n dS + \int_\Sigma \{(\rho E)_1 - (\rho E)_2\} G dS.$$

Now substitute this expression for the first term in (48) and then allow V to approach zero as in the preceeding discussion; then making use of the first shock condition (40) the resulting equation leads to the condition

$$(49) \qquad [p u_n] = \rho_1 (G - u_{1n})[E].$$

If we put $i = Jc_p T$ it follows immediately from (22) and (23) that $Jc_v T = i - p/\rho$. Making this substitution for $Jc_v T$ in the expression (25) for E we see that the relation (49) can be written

$$(50) \qquad [p u_n] + \rho_1 (G - u_{1n})\left[\frac{p}{\rho}\right] - \rho_1 (G - u_{1n})[i + \tfrac{1}{2} u_\sigma u_\sigma] = 0.$$

Now using (40), the second relation (45), and (47) it can readily be

shown that

$$[pu_n] = \{\rho_1(G-u_{1n})u_{2n} + p_1\}[u_n] ,$$

$$\left[\frac{p}{\rho}\right] = (G-u_{2n})[u_n] - \frac{p_1[u_n]}{\rho_1(G-u_{1n})} .$$

Then when we substitute these expressions for $[pu_n]$ and $[p/\rho]$ into (50) we obtain

$$(51) \qquad [i + \tfrac{1}{2}u_\sigma u_\sigma] = G[u_n] .$$

The relation (51) *is one of the forms of the third shock condition.* From the relations of §12 and the conditions (46) and (47) it follows that

$$[i] = \frac{\gamma}{\gamma-1}\left[\frac{p}{\rho}\right] ,$$

$$\tfrac{1}{2}[u_\sigma u_\sigma] = G[u_n] - \left(\frac{\rho_1+\rho_2}{2\rho_1\rho_2}\right)[p] .$$

Making these substitutions in (51) the resulting expression can be written

$$(52) \qquad \frac{p_2}{p_1} = \frac{(\gamma+1)\rho_2 - (\gamma-1)\rho_1}{(\gamma+1)\rho_1 - (\gamma-1)\rho_2} .$$

The form of the third shock condition given by (52) is usually called the *Rankine-Hugoniot relation. It holds for an ideal gas in which the viscosity and thermal conductivity are zero.* It is an interesting fact that the form (52) of the third shock condition does not involve the wave velocity G even though it is not confined to steady shock waves.

The condition (52) can be written in the following equivalent form

$$(53) \qquad \frac{\rho_2}{\rho_1} = \frac{(\gamma+1)p_2 + (\gamma-1)p_1}{(\gamma-1)p_2 + (\gamma+1)p_1} .$$

Putting $x = \rho_2/\rho_1$ and $y = p_2/p_1$ the above relation becomes

$$x = \frac{(\gamma+1)y + (\gamma-1)}{(\gamma-1)y + (\gamma+1)} = \frac{(\gamma+1)}{\gamma-1} - \frac{4\gamma}{(\gamma-1)^2y + (\gamma^2-1)} .$$

Since $\gamma > 1$, x will have its greatest value when $y = \infty$ and this maximum value of x is $(\gamma+1)/(\gamma-1)$. The least value of x occurs when $y = 0$ and is given by $(\gamma-1)/(\gamma+1)$. Hence

$$\frac{\gamma-1}{\gamma+1} \leqq x \leqq \frac{(\gamma+1)}{\gamma-1}.$$

For air $\gamma = 1.4$ approximately; for this value of γ the above inequality becomes $1/6 \leqq x \leqq 6$.

23. *The combined shock conditions.* The shock conditions which we have deduced are expressed completely by the relations (46), (47), and (52). It is a simple algebraic consequence that these relations can be put in the following equivalent form

$$(54a) \qquad [u_\alpha] = \frac{2\{\rho_1(G-u_{1n})^2 - \gamma p_1\}\nu_\alpha}{(\gamma+1)\rho_1(G-u_{1n})},$$

$$(54b) \qquad [p] = \frac{2\{\rho_1(G-u_{1n})^2 - \gamma p_1\}}{\gamma+1},$$

$$(54c) \qquad [\rho] = \frac{2\rho_1\{\rho_1(G-u_{1n})^2 - \gamma p_1\}}{2\gamma p_1 + (\gamma-1)\rho_1(G-u_{1n})^2},$$

by which the values of the quantities u_α, p and ρ on side 2 of the wave surface Σ are given directly in terms of the u_α, p and ρ on side 1 of Σ, the unit normal ν to Σ (directed from side 1 to side 2) and the velocity of propagation G of the wave surface. In the case of a stationary wave front for which $G = 0$ the relations (54) become

$$(55a) \qquad [u_\alpha] = \frac{2(\rho_1 u_{1n}^2 - \gamma p_1)\nu_\alpha}{-(\gamma+1)\rho_1 u_{1n}},$$

$$(55b) \qquad [p] = \frac{2(\rho_1 u_{1n}^2 - \gamma p_1)}{\gamma+1},$$

$$(55c) \qquad [\rho] = \frac{2\rho_1(\rho_1 u_{1n}^2 - \gamma p_1)}{2\gamma p_1 + (\gamma-1)\rho_1 u_{1n}^2}.$$

From the remark in §21 it follows that the denominator in (54a) cannot vanish at points of a shock wave. Also the numerator in (54b) cannot vanish since otherwise we would have $[p] = 0$; hence the denominator in (54c) must be different from zero as its vanishing would imply $[\rho] = \infty$ (see §19.)

24. *Change in entropy and density in passage through a shock wave.* The thermodynamic condition $d\sigma/dt \geqq 0$ for continuous flow without thermal conductivity leads to the requirement that the entropy per unit

mass σ associated with a moving particle does not decrease in the passage of the particle through a shock wave (considered as representative of the rapid variations in a flow which is actually continuous). Thus if the particle moves from side 1 to the side 2 of a shock wave Σ we must have $\sigma_2 \geqq \sigma_1$ and this implies $p_2/\rho_2^\gamma \geqq p_1/\rho_1^\gamma$ on account of the expression (32) for σ; hence

$$\frac{p_2}{p_1} \geqq \left[\frac{\rho_2}{\rho_1}\right]^\gamma . \tag{56}$$

Putting $x = \rho_2/\rho_1$ and using the Rankine-Hugoniot relation (52) for p_2/p_1 the inequality (56) becomes

$$\frac{(\gamma+1)x - (\gamma-1)}{(\gamma+1) - (\gamma-1)x} \geqq x^\gamma . \tag{57}$$

Let us now define a function y by the equation

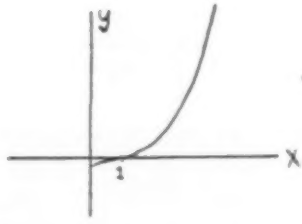
$$y = \frac{(\gamma+1)x - (\gamma-1)}{(\gamma+1) - (\gamma-1)x} - x^\gamma . \tag{58}$$

Then $y = 0$ for $x = 1$; also both dy/dx and d^2y/dx^2 vanish for $x = 1$ while

$$\frac{d^3y}{dx^3} = \tfrac{1}{2}(\gamma-1)(2\gamma-1)(3-\gamma).$$

Taking $\gamma = 1.4$ we see that $d^3y/dx^3 > 0$ for $x = 1$ so that the graph of (58) has a point of inflection at $x = 1$. The following table gives approximate values of y for the indicated values of x. The graph of this function is shown below. Condition (57) is satisfied only for values

x	y
0	-1/6
1/6	-.08
1	0
2	.11
3	1.01
4	4.54
5	19.48



of x for which $y \geqq 0$. Hence we must have $\rho_2/\rho_1 \geqq 1$. If $\rho_2 = \rho_1$ the relation (47) gives $[p] = 0$ contradicting the assumption that Σ Σ is a shock wave. It follows that $\rho_2 > \rho_1$. But $x > 1$ means that $y > 0$ and hence the inequaltiy sign holds in (57). This implies $\sigma_2 > \sigma_1$. *Hence when a fluid particle traverses a shock surface the density and entropy per unit mass associated with the particle are increased.*

If a gas increases in density when it passes through a shock wave,

the wave is called a *compression wave*. Hence shock waves, as here defined and treated, are compression waves in view of the above thermodynamic condition.

NOTES

1. O. D. Kellogg, Foundations of Potential Theory, J. Springer, Berlin, 1929. See p. 105 for definition of a *regular surface element* and p. 112 for the definition of a *regular surface*.

2. The usual proof of this relation involves the assumption of the equality of action and reaction in consequence of which the stresses T_1 and T_2 on the two sides of a surface are equal in magnitude but opposite in direction. However this can be shown by use of the condition (2). For let T_1 and T_2 be the stresses on the two sides of a regular surface element Σ (see footnote 1, *loc. cit.* p. 105). Let V be a small sphere which is divided by the surface element into volumes V_1 and V_2. Then applying (2) we have

$$\text{(a)} \qquad \frac{d}{dt}\int_V \rho u_\alpha dV = \int_S T_\alpha dS,$$

$$\text{(b)} \qquad \frac{d}{dt}\int_{V_1} \rho u_\alpha dV = \int_{S_1} T_\alpha dS + \int_\Sigma T_{1\alpha}\, dS,$$

$$\text{(c)} \qquad \frac{d}{dt}\int_{V_2} \rho u_\alpha dV = \int_{S_2} T_\alpha dS + \int_\Sigma T_{2\alpha}\, dS,$$

where S_1 and S_2 are the portions of S bounding V_1 and V_2 respectively and it is assumed that the stresses are continuous over the surface S of the sphere and on each side of the element Σ. But adding corresponding members of (b) and (c) and using (a) we obtain

$$\int_\Sigma T_{1\alpha}\, dS + \int_\Sigma T_{2\alpha}\, dS = 0.$$

Since the area of integration Σ is arbitrary it follows that $T_{1\alpha} + T_{2\alpha} = 0$. The remainder of the derivation of the relation (7) can be made by considering the stresses over the surface of a tetrahedron three of the faces of which are perpendicular to the coordinate axes. See P. Appell, Mecanique Rationnelle, vol. 3, 1928, p. 134.

3. Cf. A. E. H. Love, The Mathematical Theory of Elasticity, Cambridge, 4th ed., 1927, p. 62. Also S. Timoshenko, Theory of Elasticity, McGraw-Hill, 1934, p. 192.

4 The vanishing of the distortion tensor is evidently sufficient for $\delta\phi = 0$ in (12). To show that this condition is also necessary we have

$$\text{(a)} \qquad (D_{\alpha\beta}\nu_1^\alpha\nu_1^\beta + D_{\alpha\beta}\nu_2^\alpha\nu_2^\beta)\cos\phi - 2D_{\alpha\beta}\nu_1^\alpha\nu_2^\beta = 0$$

for arbitrary directions ν_1 and ν_2. Taking ν_1 perpendicular to ν_2 so that $\cos\phi = 0$ equation (a) gives $D_{\alpha\beta}\nu_1^\alpha\nu_2^\beta = 0$. When ν_1 has the direction of the x^σ axis and ν_2 the direction of the x^η axis, where $\sigma \neq \eta$, the latter equation gives $D_{\sigma\eta} = 0$ $(\sigma \neq \eta)$. Now choose ν_2 so that this vector does not make a right angle with any of the coordinate axes. Then taking ν_1 in the direction of the positive x' axis we have $\cos\phi = \nu_1^\alpha\nu_2^\alpha = \nu_2^1 \neq 0$. Under these conditions (a) becomes

$$\text{(b)} \qquad D_{\alpha\beta}\nu_2^\alpha\nu_2^\beta = D_{11}.$$

In a similar manner by taking ν_1 in the directions of the positive x^2 and x^3

axes, but leaving the above vector ν_2 unchanged, we obtain

(c) $D_{\alpha\beta}\nu_2^{\alpha}\nu_2^{\beta} = D_{22}$,

(d) $D_{\alpha\beta}\nu_2^{\alpha}\nu_2^{\beta} = D_{33}$.

Adding (b), (c), and (d) we find that

(e) $D_{\alpha\beta}\nu_2^{\alpha}\nu_2^{\beta} = 0$,

since $D_{\sigma\sigma} = 0$ from the equation (13) defining the deformation tensor. Now since the expression in (e) is homogeneous in the quantities ν_2^{α} this equation is seen to hold for arbitrary values of the ν_2^{α}, i.e., independently of the original condition that ν_2 is a unit vector. Hence (e) implies $D_{\alpha\beta} = 0$ or, in other words, the vanishing of the distortion tensor.

5. H. Jeffreys, Cartesian Tensors, Cambridge, 1931, p. 68.

6. The first of these limits can be inferred as follows. Under the conditions of differentiability and continuity assumed we have

$$\int_{V_1} (\rho u_a)_{,a}\, dV = \int_{S_1} \rho u_n\, dS + \int_{\Sigma} \rho_1 u_{1n}\, dS .$$

Then allowing V to shrink into the area Σ_0 the left member of this equation approaches zero and hence the limit in question is obtained. The other limit can be inferred in a similar way as well as the corresponding limits in the following sections.

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COLLEGIATE ARTICLES

Graduate Training not required for Reading

ABSTRACT ISOMORPHISM

Duane Studley

Abstract: The relation of isomorphism is investigated at some length and after consideration of the usual uses of the term and its associated notion generalization is developed. An abstract isomorphism is defined so as to contain the usual sense on specification. Some of the aspects of this consideration are new therefore some positive contribution is made.

As an aid in thought suppose I arrange a collection of piles of pebbles on the table. Actually you will have to imagine my performance of the act since I have no intention of undertaking the arduous labor of creating such an object assembly. Instead I shall look at the whole deal abstractly and place a collection of dots on the blackboard.

1	2	3	4	5	. . .	13	. . .
.	. .	. . .	. . . .	. . . . .	. . .	. . . .	. . .

The dots are not pebbles, they merely represent pebbles. In a different context I might represent an element of a group by a matrix.

Further I will perform an abstraction and represent each representation of a pile of pebbles by number symbols. In the case of the pile of thirteen pebbles I have used thirteen dots and represented these symbols by the configuration of symbols 13. As I write the symbols I simultaneously utter an auditory representation which represents the language representation and does not need translation to you. At least we agree there is a lot of representation going on around here.

The integers are no mere trite representation. They were obtained by unconscious operations of intellection, representation and abstraction, and they have the good fortune to be universals which are indispensable to thought on any level. Each integer represents a collection of classes of objects.

To further consider the notion of representation we could consider all the things a man represents. That, however, would lead us astray from the subject of this paper. Such modes of analysis are very useful and many a false argument can be laid bare with a minimum of effort. For example, Prof. Max Black in Principles of Really Sound Thinking which appeared in the March 1948 Scientific Monthly gave as maxim number three ''It all depends on who says it.'' In support of this maxim he says: ''Men are easier to classify than arguments—attend to the man, not the argument.'' Such a slight argument hardly needs refutation since one

only needs to think of the order of complexity of the objects under contemplation. A man, any man, is more complex than any argument therefore Prof. Black's argument is false.

To return, the representation on the blackboard is one-one. A set of dots corresponds to an integer. The collection of sets of dots corresponds to the collection of integers, one-one. Also the hypothetical collection of sets of pebbles stands in biunique, one-one, correspondence with the collection of sets of dots.

If we add the set of two dots to the set of three dots we get a set which is equivalent to the set of five dots. In a corresponding way when we add the integers two and three we get five. Furthermore we have assembled the parts of the notion of isomorphism. The collection of sets of dots is isomorphic to the collection of integers; that is, there is a one-one correspondence between them and under this correspondence the operation of addition is preserved. This is the basic meaning of the term isomorphism.

Thinking in a different mode and proceeding from automorphism MacDuffee defines isomorphism and then states ''Two isomorphic mathematical systems are abstractly identical.'' Such a unique statement deserves repetition here because each word contains so much meaning.

Isomorphism is a true equivalence relation. While demonstrating this instead of the usual symbol $\cong$ we will use a more general R. If we let a represent a typical integer, b the corresponding set of dots and c the corresponding pile of pebbles we can give the standard demonstration,

Reflexive: aRa for all a
Symmetric: aRb implies bRa,
Transitive: aRb and bRc imply aRc

When isomorphism is applied to different mathematical systems special conditions have to be met. For two domains to be isomorphic not only addition but multiplication too must be preserved in the biunique correspondence. The same holds true in the case of fields. Our definition is closest to the case of groups where only one operation, the group operation is involved. This group operation must of course be preserved and this one condition is the sole difference between an ordinary one-one correspondence and an isomorphism.

What I propose to do is to abstract a generalization from the isomorphism as defined. Instead of saying addition is preserved I will say the isomorphic operation remains invariant.

I define abstract isomorphism to be a relation containing a biunique correspondence under which a set of operations is invariant. If the set of operations is a null set the abstract isomorphism is merely a one-one correspondence. The question arises as to whether such a case exists. I am still investigating the structure of operations so that a statement can be made covering all operations. Even specification by which the usual senses of isomorphism are shown to be contained in this definition is itself an operation so I'll have to restrict my remarks to a simple

demonstration of inclusion.

If I specify that the set of operations consists of addition and multiplication I readily obtain the sense of isomorphism as it exists between two domains or between two fields. If I specify that successive substitution is the set of operations the term abstract isomorphism takes the sense of isomorphism between substitution groups. Since under this new definition no operation is required it can be said that an abstract isomorphism exists between matter waves and electromagnetic waves.

Foundation Research
Colorado Springs

ANNOUNCEMENT OF TWO SYMPOSIA ON NUMERICAL ANALYSIS

The National Bureau of Standards is planning two symposia on the effective utilization of automatic digital computing machinery to be held in June 1949 at the Bureau's Institute for Numerical Analysis in Los Angeles, California.

Symposium I: *Construction and Applications of Conformal Maps.* Applications of conformal maps in such fields as aerodynamics and electronics will be emphasized, with special attention devoted to the current needs of research workers.

Symposium II: *Probability methods in Numerical Analysis.* The second symposium will cover probability methods in numerical analysis, with emphasis on applications of the "Monte Carlo method" that has proven so useful in the solution of many problems in mathematical physics.

Those who would be interested in attending either of these symposia may obtain further information from Dr. J. H. Curtiss, Institute for Numerical Analysis, Los Angeles 24, California.

EQUIPARTITION OF CONVEX SETS

R. C. and Ellen F. Buck

It is well known that a line can be drawn thru any interior point of a convex set[1] dividing the set into two pieces of equal area. In *What is Mathematics*, Courant and Robbins (p. 317 et seq.) several theorems of the same type are proved. In particular, it is shown that a convex region[2] can always be divided into four equal parts by two straight cuts, and that furthermore they can be chosen to be perpendicular.

We will be concerned here with three cuts; in general they divide a convex region into seven parts. If the cuts are concurrent, six parts result. We prove that division into six equal parts is always possible, but that although some regions can be divided into seven equal parts, no *convex* region can be so divided.

Thru any point P of the boundary of a convex region there is a unique cut PP' dividing the region in half. At any point N of PP' draw a line QNQ' so that the area QNP is exactly one sixth of the whole area. (See figure 1) If N is at P', the area of $Q'P'N$ is zero; if N is at P, this area becomes one half the whole area. There is than a position of N for which $Q'P'N$ has area one sixth, and this position of N is unique since the area of $Q'P'N$ is a monotone continuous function of N on PP'. Now, draw the lines NR and NR' bisecting the remaining regions. We now have the convex set cut into six equal parts by two straight cuts, and one broken cut RNR'. Moreover, our configuration is uniquely determined by the position of the initial point P. We now show that P can be chosen so that RNR' is a straight line.

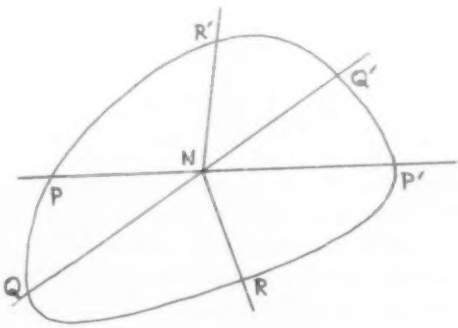

Figure 1

[1] "A set is convex if the line segment joining any two of its points lies in the set."

Move P along the boundary until it reaches the position now occupied by P'. Since the line PP' is the only line thru P' bisecting the region, P' will have moved along to take the former position of P. Again by uniqueness, Q and Q', and R and R' will have exchanged positions. Let θ be the angle RNR' measured counterclockwise; θ is a continuous function of P. The sum of the old and new values of θ is 2π; if one of the values is greater than P, the other is less. Thus, for some position of P between these extremes, θ will be exactly π; the line RNR' is then straight and the region has been divided as desired.

Where there is general intersection of the three cuts, instead of concurrency, the system has another degree of freedom. It might be expected that the lines could then be so chosen that all of the resulting regions have equal area. This is not true if the region is convex.

Suppose we have a convex region of area seven, divided by the lines EF, GH, and DI into seven regions each of area one (Figure 2).

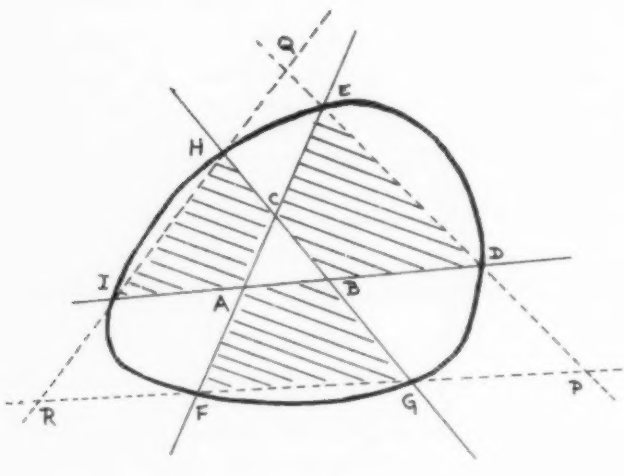

Figure 2

Starting from this assumption, we shall arrive at a contradiction. Draw the lines DE, HI and FG. The area of the shaded quadrilateral $CBDE$ is less than that of the larger curved part containing it, which is of area one. The same is true of the other two shaded quadrilaterals. Instead of the convex set we shall consider the triangle PQR and the seven parts into which it is cut by the dividing lines. The triangle ABC has area one; we can take the shaded quadrilaterals of figure 2 to have area one, and the remaining quadrilaterals $BGPD$, $CEQH$, and $AIRF$ to have area not less than one. For, otherwise we move lines DE, GF and IH outward by parallel displacements until the areas of the shaded regions are exactly one each, and this will not reduce the areas of the other three quadrilaterals.

We now observe that of all the permissable positions of the line FG

such that the area of $FGBA$ is one, $BGPD$ will have the greatest area when BG is greatest, and hence when FG passes thru A (Figure 3.) We shall prove that even in this position, $BGPD$ has area less than one, or if it is one, $AIRF$ has area zero, contradicting our assumptions.

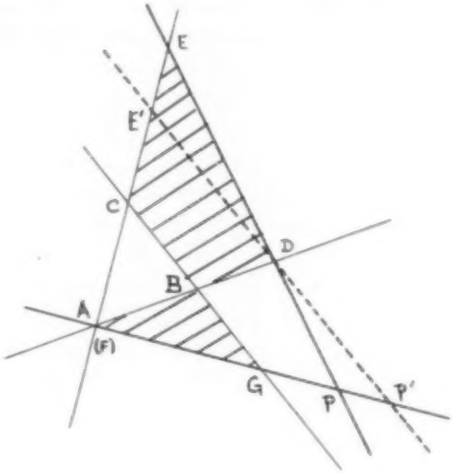

Figure 3

Suppose that ED is parallel to or converges toward CB as in Figure 3. Thru D draw $E'DP'$ parallel to CBG. Triangles ABC and ABG have area one. The quadrilateral $DECB$ has area one, and hence the area of $BDE'C$ does not exceed one, so that the area of ADE' does not exceed two. By similar triangles

$$\frac{\text{area } ADE'}{\text{area } ABC} = \frac{AD^2}{AB^2} = \frac{\text{area } AP'D}{\text{area } AGB} \leqq 2$$

so that area $APD \leqq$ area $AP'D \leqq 2$, and area $BGPD \leqq 1$. The equality sign arises only when ED is parallel to CB; otherwise $BGPD$ has area less than one, contradicting our assumption. In the extreme case, FG must pass thru A and the area of $AIRF$ is zero, instead of being not less than one. (If ED converged in the opposite direction, the argument would be the same, using the quadrilateral $CHQE$, Figure 2).

This completes the proof: *no convex set admits an equipartition by three lines.* Simple examples show that some non-convex regions can be so divided; however, the converse of the theorem is not true since examples can also be constructed which are not convex but which share the property of admitting no such equipartition.

We have actually proved more than the above. If in Figure 2, the curved pieces of the convex set labled *BCED*, *BDG*, *GFAB* and the central piece *ABC* have area one, then the piece *FAI* has area zero. As an immediate corollary of this, *if a convex set is divided into seven pieces, of which six are equal, the seventh (unequal) piece must be the central piece*, *ABC* Figure 2. If the equal pieces have area one, there is a least upper bound for the area of the seventh piece; we conjecture that the seventh piece can have area 1/8 at most. This bound is obtained by dividing a triangle with cuts parallel to the sides.

Wellesley
Harvard

A NEW LEAST-SQUARES LINE*

G. A. Baker

The usual method of fitting a straight line to a set of data as given by Snedecor (1) say, is to assume that the independent variable, x, is free of error and that the variance of the dependent variable y is uniform for all x's. In considering the relationship between white and yolk weights of birds' eggs, Asmundson *et al* (2) found that the usual assumptions were not sufficiently close to reality. A study of the actual data suggested that a close approach to reality would be obtained by considering a least-square line with the distances from the observed points to the fitted line in some direction weighted inversely as the squares of the distances of the points of intersection of the lines through the points with this direction with the fitted line from the y-intercept of the fitted line. The purpose of this note is to discuss fitting a line under these conditions.

Suppose that we assume a configuration of points (x_i, y_i) in the first quadrant and that the least $x_i > 0$. We shall assume that x_i and y_i are approximately connected by the relationship

$$(1) \qquad y = \alpha x + \beta, \ \alpha \neq 0.$$

We shall consider the distances, d_{1i}, from the points (x_i, y_i) to the line (1) taken in some direction m not necessarily perpendicular to (1). We shall consider that the weight to be attached to d_{1i} is inversely proportional to the square of the distance, d_{2i}, of the point of intersection of the line through (x_i, y_i) with the direction m from the y-intercept of (1), which is $(0,\beta)$.

We compute

$$(2) \qquad d_{1i}^2 = \frac{1 + m^2}{(\alpha - m)^2} (y_i - \alpha x_i - \beta)^2$$

and

$$(3) \qquad d_{2i}^2 = \frac{1 + \alpha^2}{(\alpha - m)^2} (y_i - m x_i - \beta)^2$$

Thus, we wish to determine m, α, and β so that

$$(4) \qquad f(m, \alpha, \beta) = \Sigma \frac{d_{1i}^2}{d_{2i}^2} = \Sigma \frac{1 + m^2}{1 + \alpha^2} \frac{(y_i - \alpha x_i - \beta)^2}{(y_i - m x_i - \beta)^2}$$

* Part of a paper presented under the title of, ''A Mathematical Model of the Relation between White and Yolk Weights of Birds' Eggs,'' at the twenty-eighth meeting of the Institute of Mathematical Statistics, June 17—19, 1947, San Diego, California.

shall be a minimum with variations in m, α, and β. Necessary conditions for such a minimum are

$$(5)\qquad \frac{\partial f}{\partial m} = \Sigma\left\{\frac{(1+m^2)(y_i - \alpha x_i - \beta)^2 x_i}{(y_i - mx_i - \beta)^3} + \frac{(y_i - \alpha x_i - \beta)^2 m}{(y_i - mx_i - \beta)^2}\right\} = 0$$

$$(6)\qquad \frac{\partial f}{\partial \alpha} = \Sigma\left\{\frac{(y_i - \alpha x_i - \beta)x_i}{(y_i - mx_i - \beta)^2} + \frac{(y_i - \alpha x_i - \beta)^2 \alpha}{(y_i - mx_i - \beta)^2(1+\alpha^2)}\right\} = 0$$

$$(7)\qquad \frac{\partial f}{\partial \beta} = \Sigma\left\{\frac{(y_i - \alpha x_i - \beta)}{(y_i - mx_i - \beta)^2} - \frac{(y_i - \alpha x_i - \beta)^2}{(y_i - mx_i - \beta)^3}\right\} = 0.$$

Now, following Deming (3) and finding the least-square line with unequal weights (not specified here but based on empirical determination) and both x and y subject to error we obtain

$$(8)\qquad y = 0.762 + 1.715x.$$

We use values from (8) as first approximations to α, β, and m. That is, replace α by $1.715 + a$; β by $0.762 + b$; and, m by $-0.583 + c$ since $m = -\frac{1}{\alpha}$ approximately where a, b, and c are now regarded as small.

Expanding (5), (6), and (7) after replacement in powers of a, b, and c and neglecting higher terms we obtain

$$(9)\qquad a = -0.155,\ b = 0.206,\ c = -0.006$$

and hence the equation

$$(10)\qquad y = 0.968 + 1.560x$$

as an estimate of the relationship between the two variables white and yolk weights of birds' eggs. The angle between the direction m and this ''best'' fitted straight line is 87° 50′ instead of 90° as assumed in deriving (8). The fit of the line (10) to the observed data appears to be good.

The conventional least-square lines measure deviations from the fitted line parallel to the x-axis, parallel to the y-axis, or perpendicular to the fitted line. The present method generalizes the direction along which deviations from the fitted line are measured. A study of the data available indicated that the weights chosen which were approximately equivalent to the square of the egg size were sufficiently close to reality to give an acceptable final result.

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University of California

TEACHING OF MATHEMATICS

Edited by

Joseph Seidlin and C. N. Shuster

This department is devoted to the teaching of mathematics. Thus articles on methodology, exposition, curriculum, tests and measurements, and any other topic related to teaching, are invited. Papers on any subject in which you, *as a teacher*, are interested, or questions which you would like others to discuss, should be sent to Joseph Seidlin, Alfred University, Alfred, New York.

ANALYTIC GEOMETRY—THE FRAMEWORK OF MATHEMATICS

Charles K. Robbins

The use of graphs dates back to the astronomer Hipparchus and probably to the ancient Egyptians. In modern times graphs have been used extensively to present statistical and scientific data. As an example consider the following table which displays simultaneous readings (to the nearest degree) on Fahrenheit (denoted by y) and Centigrade (denoted by x) thermometers.

x	−20	−10	0	10	20	30	40
y	−4	14	32	50	68	86	104

To present these data graphically, the centigrade readings are measured on a horizontal line as distances from some point O, positive to the right of O and negative to the left (Figure 1). At $x = 10$ we draw

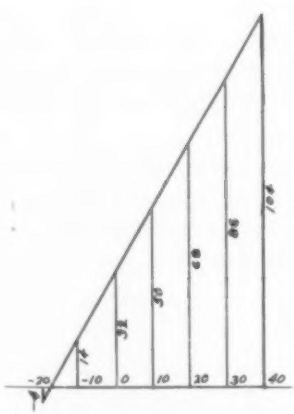

Figure 1

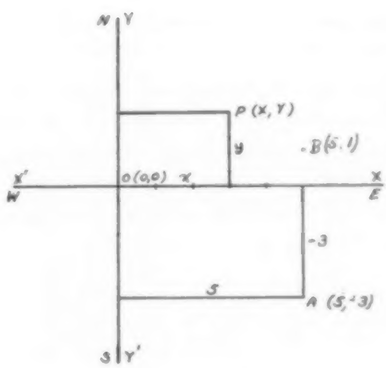

Figure 2

a line 50 units long vertically upward, at $x = 20$ a line 68 units long and so on. At $x = -20$ the line is drawn 4 units downward. We thus locate

7 points. If we were to take further readings for x between 10 and 20, say, then the readings for y would be between 50 and 68. Therefore a continuous curve drawn through the 7 points represents, at least approximately, the relation between x and y.

Often valuable information is made obvious by a graph. In this case the graph appears to be a straight line. If it is a straight line we say that the relation between x and y is *linear*. Also from the graph we can read (approximately) a value of y corresponding to any value of x. Thus if $x = 5$, $y = 41$; if $y = 0$, $x = -18$; and so on.

The study of graphs leads us to generalize the process of locating points in a plane with reference to any two mutually perpendicular lines. If you stand in a fence corner and say: ''I am 5 yards east of the North-South fence and 3 yards south of the East-West fence'', you will be understood. If you say: ''My *abscissa* is 5 and my *ordinate* is -3'', you will be using mathematical language and your less mathematical friends may regard you with the uplifted eyebrow. (If you continue the study of mathematics, you will soon discover that mathematicians tend to use ''trade'' names in order to conceal the simplicity of their concepts and impress the uninitiated.) We now change the lettering to agree with common usage (Figure 2). WE becomes $X'X$ and is called the *x-axis*; SN becomes $Y'Y$ and is called the *y-axis*. Furthermore any point P is denoted by $P(x,y)$ where x, the *abscissa*, is the distance measured from $Y'Y$ positive if in the direction of OX, negative if in the direction of OX'; and where y, the *ordinate*, is the distance measured from $X'X$, positive if in the OY direction, negative if in the OY' direction. x and y are called the *coordinates* of point P. Then A has coordinates $(5,-3)$; B, $(5,1)$; and O, $(0,0)$, called the *origin*. Thus we have made use of the real number system of algebra to set up a relation between algebra and geometry, the point being a *geometrical* concept.

To René DesCartes (1596-1650), a famous French philosopher and mathematician, is generally credited the study of plane geometry by means of algebra—namely *plane analytic geometry*. This invention also marks the beginning of modern mathematics.

To illustrate the method we now formulate two geometrical problems in terms of coordinates. The first proposes to find the distance between two points in a plane. Axes and a unit are chosen so that the coordinates of A are $(3,2)$ and B, $(7,5)$. The coordinates of C are $(7,2)$. Then (see Figure 3)

$$AC = MC - MA = 7 - 3 = 4,$$

$$CB = NB - NC = 5 - 2 = 3,$$

and

$$AB = \sqrt{AC^2 + CB^2} = \sqrt{4^2 + 3^2} = 5,$$

using the Pythagorean theorem.

To obtain a general formula for distances, we use letters and obtain

(see Figure 4)

$$AB = \sqrt{(u - x)^2 + (v - y)^2}$$

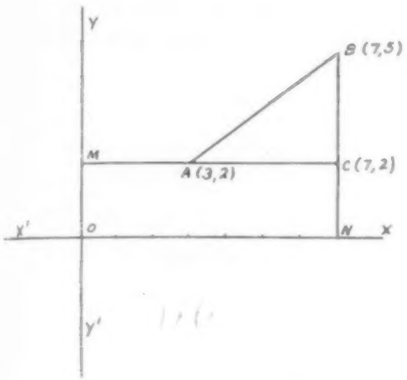

Figure 3

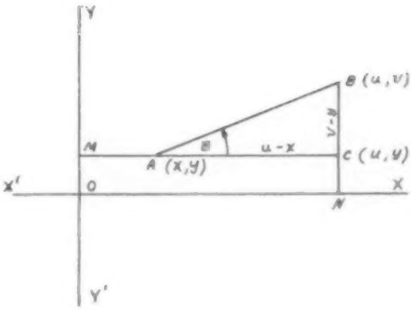

Figure 4

The second problem involves the direction of a line. This can be given by the size of the angle θ (measured counter-clockwise) which the line makes with the positive direction of the x-axis. However we shall use the slope m of AB which is defined as the ratio $\frac{CB}{AC}$. Hence $m = \frac{v - y}{u - x}$. Such ratios are in common use in measuring railroad grades, highway grades, slant of a roof and so on.

The power of the two formulas just developed is well illustrated by the following problem with which some of you may have struggled:

Two ladders of lengths 3 yards and 4 yards (Figure 5) are placed against two buildings on opposite sides of an alley as shown in the diagram. If these ladders cross at a point 1 yard above the pavement, how wide is the alley?

We choose the side of one building as OY and the pavement as OX. Then we assign coordinates to all the points involved, using letters to indicate the unknown values. Now we employ the method of algebra, namely, we seek to set up as many equations as we have unknowns. We have $OB = 4$ and $AC = 3$. Using the distance formula gives

$$\sqrt{(x - 0)^2 + (y - 0)^2} = 4 \text{ and } \sqrt{(x - 0)^2 + (0 - t)^2} = 3.$$

Also we know that the slope of OM = the slope of MB; the slope of AM = the slope of MC. The use of the slope formula gives

$$\frac{s - 0}{1 - 0} = \frac{y - 1}{x - s}; \quad \frac{1 - 0}{s - x} = \frac{t - 1}{0 - s}.$$

Thus we have expressed our geometric problem as one in algebra because

we have now to solve a set of four equations. Since we wish x, we use algebra to eliminate y, s and t and then to solve the resulting equation in x. If you are versed in algebra you can readily finish this problem.

For a further illustration we return to the graph with which we started. Assume that the graph is a straight line and let $P(x,y)$ be any point on this line (Figure 6). Then the slope of AP = the slope of AB

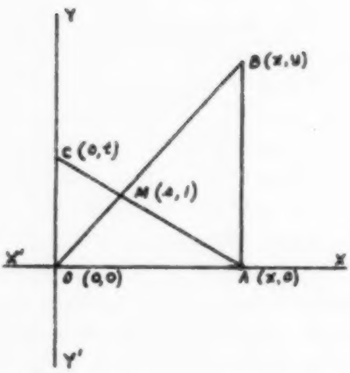

Figure 5

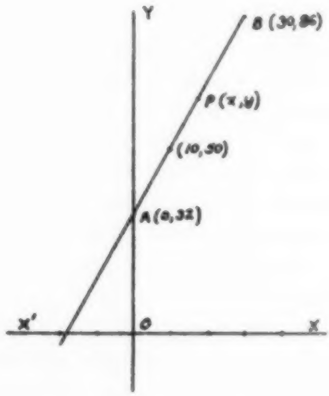

Figure 6

no matter where P is as long as it is on the line. Hence

$$\frac{y - 32}{x - 0} = \frac{86 - 32}{30 - 0}, \text{ or } y = \frac{9}{5}x + 32.$$

This means that we have found an equation which corresponds to the graph in such a way that those and only those pairs of numbers which satisfy the equation are the coordinates of points on the graph. We are now in a position to give the value of y for each value of x and vice versa. If $x = 10.5$, $y = \frac{9}{5}(10.5) + 32 = 50.9$; if $y = 0$, $x = -17\frac{7}{9}$; and so on. To express briefly the above correspondence, we say that $y = \frac{9}{5}x + 32$ is the *equation of line AB.*

In the above we can think of the straight line AB as the path traced by P if it moves always in the same direction. Suppose now that P, instead of moving along AB, moves so that it is always 5 units distant from $C(1,2)$ (see Figure 7). Then the path is a circle. Using the distance formula we get the equation of this circle, namely:

$$\sqrt{(x - 1)^2 + (y - 2)^2} = 5, \text{ or } (x - 1)^2 + (y - 2)^2 = 25$$

whence

$$y = 2 \pm \sqrt{25 - (x - 1)^2}\ .$$

Also $(x-h)^2+(y-k)^2=r^2$ would be the equation of any circle having its center at (h,k) and radius r.

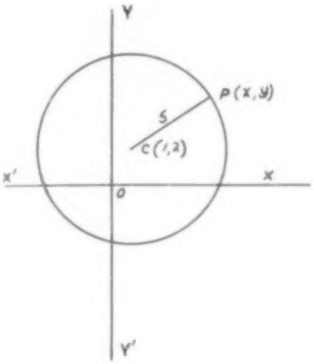

Figure 7

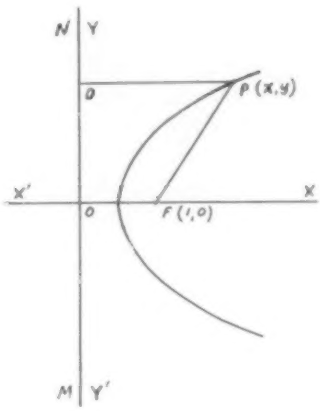

Figure 8

For both the above straight line and circle, we have obtained a relation between x and y which is such that when x is given, y can be found. In such a relation x is called the *independent variable*, y the *dependent variable*; and y is said to be a *function* of x, written $y=f(x)$. We can now state in general terms a fundamental principle of analytic geometry, namely:

$y=f(x)$ is the equation of a certain curve, if and only if the pairs of numbers which satisfy the equation are the coordinates of points which lie on the curve. Note that the word ''curve'' as used here includes a straight line. On account of the close relation between curve and equation, it is customary to abbreviate the expression ''the curve whose equation is $y=f(x)$'' to ''the curve $y=f(x)$''. Thus we say ''the line $y=\frac{9}{5}x+32$'', ''the circle $x^2+y^2=16$'' and so on.

Suppose now that the point $P(x,y)$ moves so that its distance from a fixed line MN (called the directrix) is always equal to its distance from a fixed point F (called the focus). Such a curve is called a parabola (Figure 8). To find an equation, we take MN as the y-axis and OF as the x-axis, OF being perpendicular to MN. Choose OF equal to one unit. Then for any point $P(x,y)$ on the parabola we havd $DP=FP$, whence

$$x=\sqrt{(x-1)^2+(y-0)^2}\ ,\text{ or } y^2=2x-1.$$

This is only one of the many equations which could be obtained for this parabola by various choices of axes and a unit of measure. However all these equations would be of the second degree.

We note in passing some applications of the parabola. The path of a

projectile if air resistance is neglected is a parabola. Also the new television reflector on Mount Wilson, Calif., has a shape obtained by rotating a parabola about its axis of symmetry.

It now appears that all circles and all parabolas have second degree equations. It can be shown that any second degree equation—namely $Ax^2 + Bxy + Cy^2 + Dx + Fy + E = 0$, where the coefficients are real numbers—is one of the following: circle, parabola, ellipse, hyperbola or a pair of straight lines. Examples of the latter three are, respectively: $4x^2 + 9y^2 = 36$, $4x^2 - 9y^2 = 36$, and $4x^2 - 9y^2 = 0$, with graphs as shown (Figures 9, 10, 11).

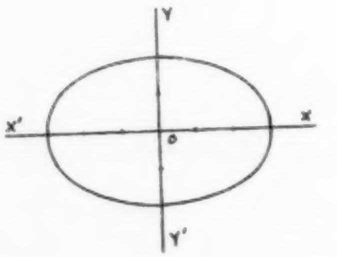

Figure 9

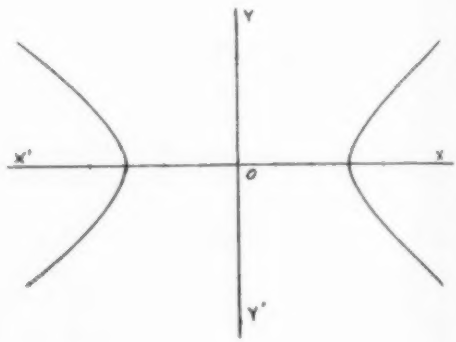

Figure 10

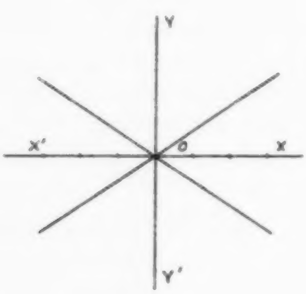

Figure 11

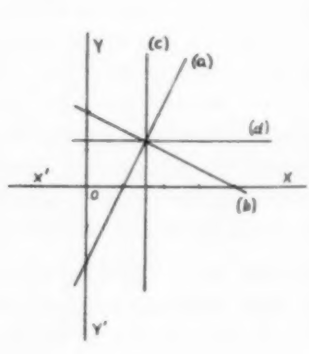

Figure 12

It seems then that some curves can be classified according to the degrees of their equations. We have noted the second degree equations above. Also, any first degree equation is a straight line; a third degree equation was classified into 78 types by Sir Isaac Newton. Such equations as these are called *algebraic* curves and have an extensive literature.

We have seen how algebra aids in the study of geometry. Conversely,

geometry illustrates some of the processes of algebra. For example, consider the problem of solving $2x - y = 2$ and $x + 2y = 4$ simultaneously. This means to find all the pairs of values which will satisfy both equations. Since the equations represent straight lines, any such pair will be the coordinates of a point which is on both lines and therefore at their intersection. Because straight lines intersect at only one point, we anticipate that the algebra will give only one pair of values. We proceed to a further geometric interpretation of the algebraic solution (see Figure 12).

$2x - y = 2$ line (a).

$x + 2y = 4$ line (b).

Eliminate y and get

$x = \frac{8}{5}$ line (c).

Eliminate x and get

$y = \frac{6}{5}$ line (d).

Thus the solution $x = \frac{8}{5}$, $y = \frac{6}{5}$ corresponds to the point $\left(\frac{8}{5}, \frac{6}{5}\right)$ which is the intersection not only of the given lines, but also of two lines parallel to the y- and x-axes respectively.

In general the algebraic solution of two equations yields the coordinates of the points of intersection of the corresponding curves.

We shall now consider a curve which is *not algebraic* and which was defined and studied by the ancient Greeks without the aid of algebra. This curve, called the *cycloid*, is the path traced by a point on the rim of a rolling wheel. The figure shows the initial position and a subsequent position of the wheel as it rolls along the level ground (taken as the x-axis). The point on the rim is initially at O and P is any subsequent position. θ is the angle through which a spoke has turned and OP is an arc of the cycloid (Figure 13). Evidently the curve will consist of a series of arches each of height 2 units, the radius of the wheel being taken as the unit.

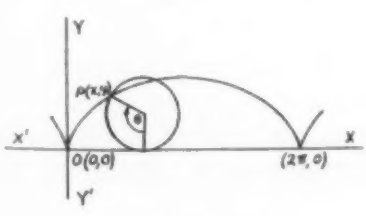

Figure 13

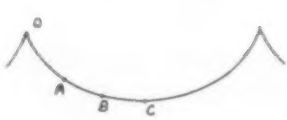

Figure 14

Using some trigonometry we get the equations

$$\left\{ \begin{array}{l} x = \theta - \sin\theta \\ y = 1 - \cos\theta \end{array} \right\}$$

For each value of θ there are values of x and y which locate a point. Hence these *two* equations serve the same purpose for which we have formerly used *one* equation. They are called *parametric* equations of the curve and θ is called a ''parameter''. To obtain *one* equation we eliminate θ and get

$$x = \text{arc cos } (1 - y) - \sqrt{2y - y^2}\ .$$

However the parametric equations are the more useful. With the aid of the *calculus* they enable us to study vehicular motion—motion of an automobile for example.

If the arch of the cycloid is inverted, then three particles sliding without friction from O, A and B at the same time will arrive at B, the lowest point, simultaneously (see Figure 14). Also a particle sliding without friction will go from O to B in less time than along any other curvd joining O and B.

The coordinates used thus far are often called *rectangular* because, if lines are drawn through the points parallel to the axes, a network of rectangles is formed. However a point in a plane can be located by means of two numbers in many other ways. We mention one of these ways.

A line $Q'Q$ is drawn through a fixed point O (called the pole) making an angle θ with a fixed line OM (called the polar axis). A distance r is measured from O along OQ to point P. in case r is positive (Figure 15). If r is negative, it is measured along OQ' to P . Thus a point P is

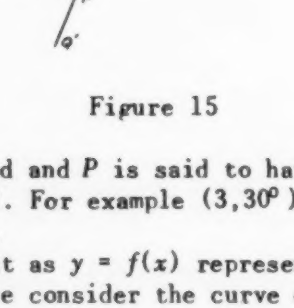

Figure 15

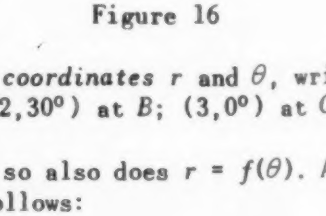

Figure 16

located and P is said to have the *polar coordinates* r and θ, written $P(r,\theta)$. For example $(3,30^\circ)$ is at A; $(-2,30^\circ)$ at B; $(3,0^\circ)$ at C and so on.

Just as $y = f(x)$ represents a curve, so also does $r = f(\theta)$. As an example consider the curve defined as follows:

Draw a circle of radius 1 and center at (1,0). Now draw *any* line OQ and extend it 1 unit to $P(r,\theta)$. Then $OP = OQ + QP$, or $r = 2\cos\theta + 1$. An arc APS of the curve is shown (Figure 16). This curve is sometimes called the trisectrix because, no matter where P is on the curve, angle AOP is one third of angle MOP.

For another example we shall find the equation of the curve traced by $P(r,\theta)$ if $\frac{OP}{DP} = e$, where DP is the perpendicular distance from a fixed line AB a units from O and perpendicular to OM, and e is a positive number less than 1. Then $OP = e \cdot DP$, whence $r = e(a + r\cos\theta)$, and $r = \frac{ae}{1 - e\cos\theta}$. For various values of a and e we get a group of curves, but it turns out that they are all ellipses (Figure 17). In particular the path of the earth around the sun is an ellipse with the sun at O.

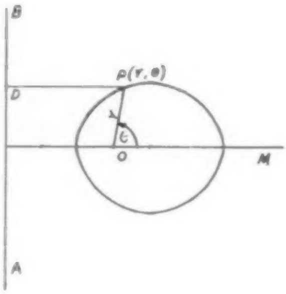

Figure 17

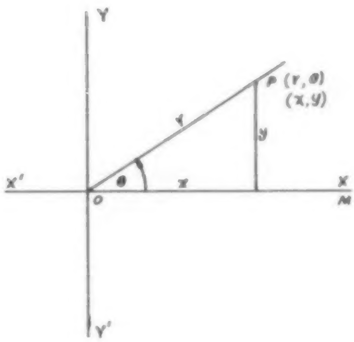

Figure 18

As we have seen, an ellipse has an equation in rectangular coordinates, but for the purposes of astronomy it turns out that the polar coordinate equation is superior. This raises the question ''What coordinate system is best?'' Since there are many systems, probably there is a best one for each geometrical problem. However the two systems we have mentioned are the simplest and are applicable to a large number of problems.

It is interesting to compare the polar coordinate equation of a curve with its equation in rectangular coordinates. To do this we let the pole and origin coincide, also the polar axis and the x-axis (Figure 18). By inspection we have $r = a$ and $x^2 + y^2 = a^2$ as equations of the circle with center at O and radius a. Evidently $r = a$ is the simpler.

On the other hand the straight line $y = \frac{9}{5}x + 32$ becomes

$r = \frac{160}{5\sin\theta - 9\cos\theta}$, a much more ''unhandy'' form. To make this

''transformation'' we note from the figure that $x = r \cos \theta$ and $y = r \sin \theta$, substitute in the rectangular equation, and do the necessary algebra.

In conclusion it should be said that a person familiar with plane analytic geometry may say that this discourse is noteworthy (if at all) for what has been left out. The reader may decide this question by studying a text on the subject or by taking a course in it. In either event he will be able to fill in the algebraic and trigonometric details which the author has so considerately omitted.

Purdue University

THE STRAIGHT LINE TREATED BY TRANSLATION AND ROTATION

Kenneth May

Transformations are usually introduced rather late in the first course in analytical geometry and appear as hardly more than a device to facilitate the graphing of conics. What should be a fundamental notion is merely a footnote. The following remarks indicate one way in which an early introduction of the equations of translation and rotation can be turned to immediate account in developing the basic theorems on the straight line. At the same time it appears that the usual proofs based on the geometrically established constancy of the slope are redundant. The nature of the graph of $Ax + By = C$ follows from the linearity of the axes without reference to the notion of slope, which can then be introduced and proved constant analytically.

Consider the equation $Ax + By = C$, where if $A = 0$, $B \neq 0$; if $B = 0$, $A \neq 0$; or $A \neq 0$, $B \neq 0$. Suppose $A = 0$. Then if $C = 0$, the graph is evidently precisely the x-axis since there and only there is $y = 0$. If $C = 0$, we translate the origin to the point $(0, \frac{C}{B})$, and the graph is now seen to coincide with the x-axis. The case $B = 0$ is treated similarly. If neither A nor B is zero, we translate the origin to the point $(\frac{C}{A}, 0)$, and the equation becomes $Ax + By = 0$. Rotation through an angle θ yields

$$(A\cos\theta + B\sin\theta)x + (B\cos\theta - A\sin\theta)y = 0.$$

Evidently the first term vanishes if $\tan\theta = -\frac{A}{B}$. Moreover, for this θ the coefficient of y cannot vanish. Hence the equation becomes $y = 0$ and its graph is coincident with the x-axis.

Thus $Ax + By = C$ is shown to be the equation of a straight line since it is in all cases congruent to an x-axis. Moreover, the argument indicates the relation between the coefficients of the equation and the slope and inclination of the line. The special forms of the equation and the constancy of the ratio $\dfrac{(y_2 - y_1)}{(x_2 - x_1)}$ follow immediately.

Carleton College

The Mathematics Magazine has its own veri-typer. It will not be very busy between April and September. We can set your papers or books for lithoprinting, at prices that will be economical for you and helpful to the magazine. Send manuscripts for estimates.

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And of course, § $\mathfrak{S}$ $\mathfrak{z}$ $\mathfrak{B}$ $\mathfrak{E}$ $\mathfrak{Y}$ $\int$ ∂ } $\surd$ Ω ξ ψ complete mathematical symbols.

CURRENT PAPERS AND BOOKS

Edited by

H. V. Craig

This department will present comments on papers previously published in the MATHEMATICS MAGAZINE, lists of new books, and book reviews.

In order that errors may be corrected, results extended, and interesting aspects further illuminated, comments on published papers in all departments are invited.

Communications intended for this department should be sent in duplicate to H. V. Craig, Department of Applied Mathematics, University of Texas, Austin 12, Texas.

Comment On Two Papers By P. A. Piza. By H. W. Becker

I wish to congratulate Senor Piza upon his elegant theorems in escalator number theory, and to flash some sidelights.

Where Ω and Ω' are any two operations, $s\Omega z = s\Omega' z$ is an operational equation. The escalator numbers A_n are then solutions of an op. eq. between + and × (1). These with exponenting are the first three of an infinite series of operations, which have been variously defined (2-6). These operations each have an inverse (as / is the inverse of ×) and a dual (as $||$ is the dual of +, thus $s||z = sz/(s + z)$ is the dual of $s + z$). One innovator (7) proposed that we write the third operation $s^z = s\$z$, because ''money is power''; and its inverse $s^{1/z} = s¢z$, because ''money is the root of all evil''. Howsoever that may be, + = \$ would be a concise instruction to solve an op. eq. $s + z = s\$z$, etc.

The fundamental equation of escalator number theory is ''$A_n + A_{n+1} = A_n \times A_{n+1}$'', whence ''$1 = A_nA_{n+1}/(A_n + A_{n+1}) = A_n||A_{n+1}$''. Or in electrical parlance, ''The joint impedance of two successive cobasic escalator numbers in parallel is an impedance match of unity''.

The impulse behind this theorem goes back to the very dawn of Number Theory. That $1 + 2 + 3 = 1 \times 2 \times 3$ so impressed the ancients, they called 6 the first of the 'perfect numbers'. They went so far as to inscribe the related 'amicable numbers' on pellets, and eat them for an aphrodisiac (8).

Being with Piza a connoisseur of recurrence gems, I wish to exhibit further references for his excellent development of the Kummer Numbers—called cumulative numbers, and similarly employed (9). They perhaps originated in a summation of series by Euler (10) in a letter to Stirling, dated ''27 Julii 1738''.

They have a combinatory interpretation (11), ${}_tK_c$ is the number of

permutations of t letters with $c - 1$ inversions. *E.g.*, *abc* has no inversions, *cba* has two, and *bac* has one inversion of standard order between consecutive elements. They also enumerate ascending runs and triangular permutations (12). They have yet a different type of interpretation: ${}_tK_c$ is the number of words of t letters, c different, such that the p^{th} position may be occupied by any of the first p letters of the alphabet. Thus *aaa* is a member of ${}_3K_1$, *aac* of ${}_3K_2$, and *abc* of ${}_3K_3$.

The expositor's enthusiasm for the summational advantages of the D formula is shared by practical statisticians (13-16), among whom it has largely superseded Bernouilli's method (17)—which discourse will however remain long a classic in the literature of jubilation, and symbolic polynomials.

Devotees of both music and mathematics may be interested in the combinatory interpretation of ${}_tD_c = \Delta^c O^t$, the number of musical cadences of t notes, c different. Thus that most celebrated example *bbba*, the opening bar of Beethoven's Fifth Symphony, is a member of ${}_4D_2 = 14$. For $t < 6$, these may be written down by permuting all the rhyme schemes of page 23, XXII, this Magazine, since ${}_tD_c = c! \cdot {}_c\mathfrak{S}_t$.

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Omaha, Nebraska

Tables of Bessel Functions of Fractional Order, Vol. I., Columbia University Press, N. Y., 1948. 413 pp. Price $7.50. The Computation Laboratory, National Applied Mathematics Laboratories, National

Bureau of Standards.

This is another of the series of tables produced by what was originally known as the NYMTP (New York Mathematical Tables Project) which began under the auspices of the Works Progress Administration for the City of New York. The first table appeared in 1939. Work continued with the aid of funds from the Office of Scientific Research and Development and was operated by the National Bureau of Standards. The project has now become one of the National Applied Mathematics Laboratories. However, it is to miss the whole truth if one looks only at the agency from which the table has emerged and forgets the human side of the picture, the men behind the table: They are Arnold N. Lowan, now chief of the laboratory, and M. Abramowitz, G. Blanch, A. Hillman, W. Horenstein, M. Karlin, J. Laderman, I. Rhodes, H. E. Salzer, I. Stegun of the technical staff. It is out of their drive, hopes and feelings that tables and programs for tables arise and have their being.

In the growing field of mathematical technology, mathematical tables and other aids to computation assume central importance. Only those functions that are usable without tabulation or that are adequately tabulated can be used in the solution of practical problems, requiring numerical answers. Numerical results are the basis for comparison of theory and measurement and for the prediction of the performance of a proposed device. When a function has been tabulated, *e.g.*, sin x, cos x, log x, etc., it becomes available for the expression of the solution of a mathematical-technical problem in ''closed'' form. Almost all problems in mathematical technology which involve the deduction of the consequences of laws of physics, also involve approximations and simplifying assumptions. In many cases, the factor which governs these simplifying assumptions is the existence of known or tabulated functions to which the assumption leads. Even the use of Fourier series and power series can be viewed as an approximation which yields the answer as a linear combination of trigonometric functions or as a polynomial.

The foreword by Schelkunoff, the introduction by Abramowitz and the bibliography are important parts of the table. The foreword indicates the differential equations which are solvable in terms of the Bessel functions of fractional order. It is also pointed out that among the problems which lead to these differential equations are wave propogation in stratified media with a constant gradient in the index of refraction and a non-uniform transmission line. The introduction discusses the mathematical properties of the functions tabulated, differential equations reducible to Bessel's equation, zeros, interpolation, Bessel functions considered as functions of their order (by Salzer), method of computation, and the preparation of the manuscript. The bibliography gives thirty references: *a*) tables and texts, 1–11, *b*) articles relating to the theory of Bessel functions, 12–22, *c*) texts and articles relating to physical applications, 23–30.

The technical details of the table are:

1. Tables of $J_\nu(x)$

ν	x
$-3/4$, $-2/3$	$[0(.001)0.9(.01)25;\ 10D]$
$-1/3$, $-1/4$	$[0(.001)0.8(.01)25;\ 10D]$
$1/4$, $1/3$	$[0(.001)0.6(.01)25;\ 10D]$
$2/3$, $3/4$	$[0(.001)0.5(.01)25;\ 10D]$

2. Tables of $A_\nu(x)$ and $B_\nu(x)$:

$$J_\nu(x) = A_\nu(x)\cos(x - \pi\nu/2 - \pi/4) - B_\nu(x)\sin(x - \pi\nu/2 - \pi/4)$$

$$\pm\,\nu = 1/4,\ 1/3,\ 2/3,\ 3/4$$

$$x = [25(.1)50(1)500(10)5{,}000(100)10{,}000(200)30{,}000;\ 10D]$$

3. Zeros of $J_\nu(x)$

4. Tables of $L_\nu(\mu)$—For interpolating in the ν-direction.

5. Other auxiliary tables.

There is no question of the reviewer judging the merits of the tables; they are excellent tables and fill an important need. Those who are interested in the general subject of tables will find two sources helpful. The first source is the journal ''Mathematical Tables and other Aids to Computation.'' The second source is the book ''An Index of Mathematical Tables,'' by A. Fletcher, J.C.P. Miller, and L. Rosenhead, publisher McGraw-Hill, N.Y., 1946.

Concord, Massachusetts — Nathan Grier Parke III

Fundamentals of Statistics. By J. B. Scarborough and R. W. Wagner. Boston, Ginn and Co., 1948 7 plus 145 pages. $2.40

Books in statistics written by statisticians usually turn out well while those written by non-statisticians usually turn out badly. Unfortunately practically all elementary books in statistics belong in the latter category; the present volume was written by two mathematicians at the U. S. Naval Academy. Apparently they made the mistake of letting themselves be guided by other elementary books in the field.

The book is intended as a textbook for a brief course in statistics for students who have had calculus. It is clearly written and contains

a wealth of exercises. Considerable material is packed into its 108 pages: frequency distributions, the first four moments, correlation, regression, binomial and normal distributions, Gram-Charlier series, and a few large sample tests of significance. Also there is an appendix on combinatorial probability, and tables of the normal distribution and and its first four derivatives.

The trouble with the book is in its selection of material. The main problem of statistics is uncertain inference: sampling theory, estimation theory, and tests of hypotheses. The book relegates all these fundamental problems to a brief final chapter. It chooses instead to emphasize the frequency tally, the class interval, the miscellaneous means, skewness and kurtosis, computational details, and the Gram-Charlier series. These topics are included at the expense of Student's distribution, the Chi-square distribution with all its many uses, and the most powerful technique the statistician has—the analysis of variance. The student who surveys statistics through these pages will not see it as the exciting subject it is, but merely as an insane accounting system apparently bent on reducing any number of figures to four figures.

Although it now appears utterly useless to do so we are duty-bound to observe some of the ancient errors reproduced in this book. First of all there is the nonsense about skewness and kurtosis. Statisticians have been poking fun at skewness and kurtosis in their classrooms for years. They have buried skewness and kurtosis in their journals time and time again. Yet here they are again, not only hale and hearty, but of sufficient stature to be included in a severely limited selection of topics.

Another thing statisticians like to exercise their sarcasm on is the band about a regression line bounded by parallel lines. This text does not deprive them of that opportunity.

A favorite classroom sermon has to do with the sin of equating parameters to their estimates. The authors do just that on page 105. To me, page 105 is the saddest part of the book for here the authors commit two mathematical crimes. The second one is in leading the student to believe that an unbiased estimate of a function of a parameter may be obtained by using the same function of an unbiased estimate of the parameter.

Of course every book has its little errors, and by emphasizing them a reviewer can give a bad impression of a good book. But the errors mentioned here are not of that category. Statisticians will immediately deduce that authors who can make such errors do not read the statistical journals, have not read Fisher's books, have not read the Neyman and E. S. Pearson papers, have probably not even read Karl Pearson. In short they have attempted to write a textbook in a field about which they are ill-informed.

A. M. Mood

The authors will reply in the May-June issue of this Magazine—*Ed.*

PROBLEMS AND QUESTIONS

Edited by

C. G. Jaeger, H. J. Hamilton and Elmer Tolsted

This department will submit to its readers, for solution, problems which seem to be new, and subject-matter questions of all sorts for readers to answer or discuss, questions that may arise in study, research or in extra-academic applications.

Contributions will be published with or without the proposer's signature, according to the author's instructions.

Although no solutions or answers will normally be published with the offerings, they should be sent to the editors when known.

Send all proposals for this department to the Department of Mathematics, Pomona College, Claremont, California. Contributions must be typed and figures drawn in india ink.

SOLUTIONS

No. 21. Proposed by *Julius Sumner Miller,* Dillard University

A plane is inclined to the horizontal at an angle B. At the foot of this plane a particle is projected with velocity V at an angle A with the plane. Find the condition for maximum range.

Solution by *Howard Eves,* Oregon State College.

This is an old problem; see, *e.g.,* J. H. Jeans, *An Elementary Treatise on Theoretical Mechanics,* pp. 209-210. It is there shown that

$$A = 45° - \frac{B}{2}$$

That is, to get maximum range up the inclined plane we must project in the direction which bisects the angle between the inclined plane and the vertical. This maximum range has the value

$$R_{max} = \frac{V^2}{g(1 + \sin B)}.$$

Solved also by Leo Moser, Winnepeg, Canada.

No. 27. Proposed by *C. W. Trigg,* Los Angeles City College.

In a square of side "a" a right triangle is inscribed with one acute angle at a corner of the square and the other two vertices on the sides of the square non-adjacent to that corner. If one of the legs of the triangle is equal to the other plus the side of the square, find the longer leg of the triangle.

Solution by *C. D. Smith,* University, Alabama

I first solve the more general problem for a rectangle with sides a and b. ($a > b$). The square is a special case. The right triangle has vertex of the right angle along side b with legs c and d. ($c > d$).

Let x be the third side of the triangle formed by c and a. The angle formed by c on a is equal to that formed by d on b, hence the relation, $c/a = d/(b - x)$. Also $x^2 = c^2 - a^2$. Eliminate x and apply the given condition $d = c - a$. Using the relation $r = b/a$ we have by obvious reductions the equation

$$(1) \qquad c^4 - c^2a^2(2 - 2r + r^2) + 2ca^3(1 - r) - a^4 = 0.$$

A positive value of c greater than a is a solution. For $r = 1$ we have the special solution for a square.

$$(2) \qquad c^4 - c^2a^2 - a^4 = 0$$

The positive root of (2) is $c = \frac{a}{\sqrt{2}}\sqrt{(1 + \sqrt{5})}$.

Solved also by the proposer, and *Adrian Struyk*, Patterson N. J.; *W. B. Clarke*, San Jose; and *Francis L. Miksa*, Aurora, Illinois.

PROPOSALS

34. Proposed by *Victor Thébault*, Tennie, Sarthe, France.

Find a number of four digits such that its square ends with the same four digits in the same order. Show that its cube and its fourth power have the same property.

35. Proposed by *Victor Thébault*, Tennie, Sarthe, France.

A plane P divides the volume of a sphere into two parts V and V' and determines two spherical segments of areas S and S'. If it is known that $S/S' = k$, calculate the ratio V/V' in terms of k.

36. Proposed by *Julius Sumner Miller*, Michigan College of Mining and Technology.

A gun can put a projectile to a height R/n, R being the radius of the Earth. Assuming the variation in gravitational force with altitude, find the area commanded by the gun.

Problems 18 and 19 in Vol. 22, No. 1 were proposed by *Julius Sumner Miller* rather than by *Julius Sumner* as there indicated. Problem 18 as solved in Vol. XXII No. 2 should have been credited to *K. L. Cappel*.

MATHEMATICAL MISCELLANY

Edited by

Marion E. Stark

Let us know (briefly) of unusual and successful programs put on by your Mathematics Club, of new uses of mathematics, of famous problems solved, and so on. Brief letters concerning the MATHEMATICS MAGAZINE or concerning other "matters mathematical" will be welcome. Address: MARION E. STARK, Wellesley College, Wellesley, 81, Mass.

NBS Interim Computer

"Design and development work supported by the Air Force is well under way at the National Bureau of Standards for the construction of a small-scale electronic computing machine to be used until the several large-scale machines now being built become available. The new high-speed machine, to be known as the NBS Interim Computer, will perform as substantial portion of the computation work of the Bureau's laboratories, solving many problems until recently considered impossible of solution. It will also aid in computing machine development at the Bureau and will provide important training and operational experience for personnel of those agencies that plan to operate the more complex electronic computers as soon as their construction is complete."

"The National Bureau of Standards is now engaged in an extensive computer program, undertaken in cooperation with the Office of Naval Research, the Bureau of the Census, the Department of the Army, and the Department of the Air Force. This program involves the research, design, and development work necessary to produce electronic machines that will perform, upon instruction, predetermined sequences of calculation running into the thousands of operations without the intervention of human operators. The result will be the solution in a few hours of complex problems in atomic physics, ballistics, and aero-dynamics which cannot now be solved except by simplifying assumptions and thousands of man-days of work. The rapidity with which numerical data can be handled, classified, and analyzed will also be correspondingly increased."

National Bureau of Standards — Technical Report 1321
Washington 25, D. C.

Sums of Two Squares

Beginners in algebra are taught the factors of $a^2 - b^2$, but are usually impressed with the fact that the *sum* of two squares cannot be factored. Certainly $a^2 + b^2$ has no real factors of the type considered in elementary algebra. But $(a^2)^2 + (2b^2)^2$ has the factors

$$(a^2 + 2ab + 2b^2)(a^2 - 2ab + 2b^2).$$

There are many cases in which a numerical integer, found as the sum of two squared integers, has integral factors. The search for such integers raises some interesting questions.

The well known formula which gives integer sides for right triangles,

$$(a^2 - b^2)^2 + (2ab)^2 = (a^2 + b^2)(a^2 + b^2)$$

gives two equal factors for a sum of two squares.

If we ask what integers are the sum of two integers squared in two different ways, we start with $a^2 + b^2 = x^2 + y^2$, whence $a^2 - x^2 = y^2 - b^2$, and

$$(a + x)(a - x) = (y + b)(y - b)$$

This equation can be satisfied by taking the product as $4PQ$ and making

$$a + x = 2PQ \qquad y + b = 2P$$
$$a - x = 2 \qquad y - b = 2Q$$

whence

$$a = PQ + 1 \qquad b = P - Q$$
$$x = PQ - 1 \qquad y = P + Q$$

So we have two cases of the sum of two squares and the factors

$$(PQ \pm 1)^2 + (P \mp Q)^2 = P^2Q^2 + P^2 + Q^2 + 1 = (P^2 + 1)(Q^2 + 1)$$

For example, if $P = 2$, $Q = 3$, this formula gives

$$7^2 + 1^2 = 5^2 + 5^2 = 5 \times 10 = 50$$

50 is the smallest number that can be expressed as the sum of two squares in two different ways.

To make sure that a and b differ from x and y, note that $a \neq x$, and if $a \neq y$, $PQ + 1 \neq P + Q$, which gives $(P - 1)(Q - 1) \neq 0$, so it is only necessary that neither P nor Q equals 1.

The formula $P^2Q^2 + P^2 + Q^2 + 1$ may be taken as the sum of two squares in a third way, if we impose an additional condition. This formula ends in a square, 1, and $P^2Q^2 + P^2 + Q^2$ is a square if $(P^2)^2 = 4(P^2Q^2)Q^2$, that is if $P = 2Q^2$. Using this value we get

$$4Q^6 + 4Q^4 + Q^2 + 1 = (2Q^3 + Q)^2 + 1^2 = (4Q^4 + 1)(Q^2 + 1)$$

This gives us another case of factors for the sum of two squares.

This product, $(4Q^4 + 1)(Q^2 + 1)$, provided $Q \neq 1$, gives a number which can be separated into the sum of two squares in three different ways: the original two ways are

$$(2Q^3 \pm 1)^2 + (2Q^2 \mp Q)^2$$

and the third way is

$$(2Q^3 + Q)^2 + 1^2$$

For example, if we take $Q = 2$, we get

$$(16 \pm 1)^2 + (8 \mp 2)^2 = (16 + 2)^2 + 1^2 = 65 \times 5 = 325$$

325 is the smallest integer that can be separated into the sum of two squares in three different ways.

Tufts College William R. Ransom

More news from France, sent us by Colonel Byrne. Professor Gaston Julia has been elected Vice President for 1949 of the Académie des Sciences. Three volumes of the collected works of Poincaré have been published, while seven more await publication. Inflation in France has reached such a state that twenty million francs will be needed for that publication.

New York, January 27: A feature of the annual meeting of the Duodecimal Society of America, held tonight at the Gramercy Park Hotel in New York City, was the explanation of a new, mathematically-based notation for music by Velizar Godjevatz. Mr. Godjevatz, who has studied music at Oxford, Grenoble, and in Berlin, pointed out the illogic of applying to our present 12-tone tempered system the staff of five lines and the concepts of the octave and a 7-tone system.

"It should be borne in mind," he said, "that our present musical notation was conceived for another musical system in use centuries ago. When Werckmeister definitely established our 12-tone equidistant system, and Bach drew the necessary conclusions in publishing his *Well Tempered Clavichord* in 1722, the notation was not changed to accomodate the practical results obtained by embracing this new system."

He proposed the "Notation Godjevatz" in which each of the twelve tones within a Region (his term for the sound range of an octave) has separate representation on a staff of seven lines, with the middle line eliminated for easier scanning except when a note falls upon it. This staff requires neither sharps nor flats, and is proposed as greatly simplifying the learning and the performance of music, both for the amateur and the professional.

Mr. Godjevatz explained the mathematical relationships among musical tones, calling a notation that would make these relationships plain the most important problem in music today. "It would be advisable to employ for our dodecaphonic musical system the duodecimal number system," he declared. "Every tone could be represented by one digit only, and this would be its relative pitch. The adoption of the duodecimal musical notation would create a sort of musical arithmetic. This would encourage more people to learn to play an instrument, people who otherwise would not expend the time and effort required to master a good reading knowledge of music."
